DAVIDSON LABORATORY

STEVENS INSTITUTE OF TECHNOLOGY
CASTLE POINT STATION
HOBOKEN, NEW JERSEY

Report 1313

WAVE-ICE INTERACTION

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ABSTRACT

Three models are examined to study the transmission of ocean waves through an ice-field. In each case the effect of ice thickness, water depth, and the wavelength and angle of incidence of the incoming ocean wave is considered. In Model I the ice is assumed to consist of floating non-interacting mass elements of varying thickness and the shallow-water approximation is utilized to simplify the equations. A simple cosine distribution varying in one direction only is assumed. In Model II the mass elements, of constant thickness, interact through a bending stiffness force so that the ice acts as a thin elastic plate. The mass elements are connected through a surface tension force in Model III so that the ice is simulated by a stretched membrane. In both Models II and III the full linearized equations are solved. Because of the complexity of the resulting analysis, calculations of the reflection and transmission coefficients, and the pressure under the ice, are made in Model II on the basis of the shallow water approximation.

KEYWORDS

Hydrodynamics

Wave-Ice Interaction

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NOMENCLATURE

Unless indicated, equation numbers refer to Part II of the report.

A	constant defined by Eq. (3.5); constant defined by Eq. (5.22)
$A_i (i = 0,1,2)$	constants defined by Eq. (5.25)
$A_{i,j} (i,j = 1,2)$	constants defined by Eq. (3.56)
a	constant in Mathieu's equation, defined by Eqs. (2.7) and (3.2) of Part I
$\pm ia_o$	pure imaginary roots of Eq. (2.1)
$\pm a_n (n = 1,2,3...)$	real roots of Eq. (2.1)
$\pm a_o'$	$= \pm(a_o^2 - k^2)^{\frac{1}{2}}$
B	constant defined by Eq. (2.2)
$\pm ib_o$	pure imaginary roots of Eq. (2.2)
$\pm b_n (n = 1,2,3...)$	real roots of Eq. (2.2)
$\pm b_o'$	$= \pm(b_o^2 - k^2)^{\frac{1}{2}}$
C	path of integration for the integral in Eq. (3.33)
C', C''	deformed paths of integration
C_1, C_2	constants defined by Eq. (3.30)
$c = \sqrt{gH}$	shallow water wave velocity
$c_i (i = 1,2,3,4)$	constants defined by Eq. (3.19)

$\pm c_o$, $\pm \bar{c}_o$	complex roots of Eq. (2.2)
$\pm c_o'$	$= \pm (c_o^2 + k^2)^{\frac{1}{2}}$
$\pm \bar{c}_o'$	$= \pm (\bar{c}_o^2 + k^2)^{\frac{1}{2}}$
D	$= \dfrac{Eh^3}{12(1-\nu^2)}$; strip in the complex α-plane, $-k < \tau < 0$
D_+	region $\tau > -k$ in the complex α-plane
D_-	region $\tau < 0$ in the complex α-plane
E	Young's modulus for ice
$F_1(\alpha)$	$= J(\alpha) - \dfrac{iAK_+(-b_o')}{\alpha + b_o'}$
$F(x,y)$	defined by Eq. (1.8) of Part I
$G(y)$	defined by Eq. (2.6) of Part I
g	acceleration due to gravity
$g(\beta)$	defined after Eq. (3.53)
H	depth of water
$H(x)$	solution of Eq. (1.10) obtained from separation of variables
h	constant ice thickness
h_o , h_1	mean and fluctuating ice thicknesses in Part I
$h(x,y)$	variable ice thickness in Part I

I	coefficient of incident potential
i	$= \sqrt{-1}$
$J(\alpha)$	$= C_1\alpha + C_2$
K	wave number for incident waves in deep water
K_o	wave number for incident waves in shallow water
K_o'	$= (K_o^2 - k^2)^{\frac{1}{2}}$
$K_\pm(\alpha)$	functions regular and non-zero in $D_\pm$ respectively, defined by Eq. (3.25) and evaluated in Appendix B
k	wave number for undulations in the ice (Part I) wave number for x-component of incident waves
$k_n (n = 0, 1, \ldots 5)$	roots of the equation $Mk_n^6 + (1-L)k_n^2 = K_o^2$
k_n'	$= (k_n^2 - k^2)^{\frac{1}{2}}$
L	$\dfrac{\rho_i \omega^2 h}{g}$
M	$D/\rho g$
P	coefficient in pressure fluctuation term, Eq. (5.38)
$\mathcal{P}$	non-dimensional value of the pressure amplitude
p	water pressure
p_o	atmospheric pressure
q	constant in Mathieu's equation defined by Eq. (2.8) of Part I

R	coefficient of reflected potential
$\mathcal{R}$	reflection coefficient
r	$= [y^2 + (z-H)^2]^{\frac{1}{2}}$
S	$= T_s/\rho g$
s	$= \rho_i/\rho$, specific gravity of ice
T	coefficient of transmitted potential
$\mathcal{T}$	transmission coefficient
T_s	suriace tension force
t	time
x,y	horizontal space co-ordinates
z	vertical space co-ordinate
α	$= \sigma + i\tau$, complex Fourier transform variable
β	constant defined by Eq. (3.10)
β'	$= (\beta^2 - k^2)^{\frac{1}{2}}$
γ	$= (\alpha^2 + k^2)^{\frac{1}{2}}$
ϵ	$= \frac{1 + \sqrt{3}i}{2}$
η	defined by Eq. (2.6)
θ	angle between direction of incident wave and normal to ice field
θ_T	angle between direction of transmitted wave and normal to ice field

θ_{crit}	critical angle at which complete reflection occurs
λ	wavelength of incident wave
λ_{crit}	critical wavelength above which all waves penetrate the ice
λ_{min}	minimum wavelength below which effectively no penetration occurs (Part I)
ν	Poisson's ratio for ice
ξ	time dependent surface elevation
ρ	density of water
ρ_i	density of ice
σ	real part of complex variable α
τ	imaginary part of complex variable α
$\Phi(x,y,z,t)$	time dependent velocity potential
$\varphi(x,y,t)$, $\emptyset(y,z)$	shallow water velocity potential defined by Eq. (1.5) in Part I; defined by Eq. (1.9) in Part II, respectively
$\Psi(\alpha,z)$	Fourier transform of $\psi(y,z)$
Ψ_+ , Ψ_-	half-range Fourier transforms defined by Eqs. (3.14) and (3.15)
$\psi(y,z)$	$= \varphi(y,z) - e^{ib_o'y} \cosh b_o z$
ω	wave frequency

DIVISION OF STUDY

The study is divided into two distinct parts. Part I consists of Model I, the propagation of waves through ice of variable thickness in shallow water. This part is complete in itself, containing the analysis, some numerical results, and a qualitative discussion of the importance of the various parameters in predicting wave transmission and reflection. The main part of this work was carried out by Professor T. V. Davies, Visiting Scientist, Davidson Laboratory, September 1965 to July 1966.

Part II contains Models II and III, where the ice is assumed to have a flexural stiffness and a surface tension-type force, respectively. For Model II, numerical results for transmission and reflection coefficients, together with the amplitude of the pressure fluctuation on the bottom under the ice, are given, together with a discussion of the results.

Each part is complete in itself and may be read separately.

INTRODUCTION

This report forms the theoretical part of a combined theoretical and experimental study[15] into the effect of water waves on an ice-field in water of finite depth.

Wave transmission and reflection in finite and infinite depths of water, partially ice-covered, have been the subject of a number of theoretical studies[1,2,3,4,5,6] in contrast to the scarcity of experimental ones.[7,8] The theoretical studies have been dominated by the basic assumption that the sheet of ice can be represented by a semi-infinite rigid sheet, or by a sheet composed of non-interacting floating point masses. So far, theory reveals that the transmission of propagating undamped waves under the ice depends upon the assumed surface condition, the ice thickness, the angle of incidence of the incident waves, the wavelength of such waves, and the depth of water.

Heins[1] has assumed the ice to be a continuous rigid sheet without movement, extended over a semi-infinite region of finite depth; he studied the water wave-ice interaction under these conditions. Peters[2] assumed that the ice consisted of broken pieces with no interaction, i.e., neither stiffness nor elasticity, extending over a semi-infinite region on the surface of an infinite depth of water; he found that there is a critical incidence frequency which determines whether or not there will be a damped transmitted wave after the interaction. Keller and Weitz[3] have analyzed the same problem, but with water of finite depth. Shapiro and Simpson[4] have made a numerical study of the above reference which shows that water waves entering an ice field are damped exponentially with increasing ice thickness. Keller and Goldstein[5] have considered the reflection of water waves from a region covered by floating matter with and without surface tension; they have solved the wave-ice interaction problem for an arbitrary incidence angle by utilizing the shallow water theory. Their study reveals the importance of the incidence angle as well as of the degree of stiffness of the floating material. Perhaps the most realistic model for the transmission of water waves by large floes is given by Stoker[6] although his motivation was different. He was concerned with the effectiveness of a floating beam having a known flexural stiffness, as a breakwater. The treatment is restricted to two dimensions in the sense that the incident waves enter the ice normally, and the shallow water approximation is utilized.

Finally, the observations of Robin[7,8] on wave propagation through ice fields are considered to be fundamental in which he demonstrates the existence of a higher cut-off wavelength depending on the size of floes, above which no attenuation of water waves occurs, and a lower cut-off wavelength below which the incident wave is completely absorbed.

The present study consists of three distinct models. In the first model the ice is assumed to be made up of floating non-interacting elements of varying mass density distribution. The shallow water approximation is used and a simple thickness distribution is considered to simplify the analysis. A qualitative discussion is made of the importance of the variation in floe thickness, the angle at which the incident wave approaches the ice-field, the wavelength of the incident wave, and the depth of water. It is shown that transmission of a given incident wave at a given incidence angle is critically dependent upon the parameters of the problem, but that waves which are long enough will always be transmitted into the ice. In addition, some computations are made which indicate in effect a wavelength below which no incident waves penetrate the ice. These results confirm the observations made by Robin.[7]

In Model II the ice is assumed to have a constant thickness h, but is allowed to bend to permit the transmission of water waves. This corresponds to Stoker's model, but in this case waves may be incident on the ice from any angle and the problem is solved for any depth of water H. Because of the complexity of the analysis, numerical computations are made only under the assumption of the shallow water theory. It is shown that a critical incidence angle exists for each wavelength above which the incident wave is completely reflected by the ice-field, without transmission taking place. In addition there exists a critical wavelength above which all incident waves, at any incident angle, penetrate the ice-field as unattenuated transmitted waves of reduced amplitude. Extensive calculations are made which emphasize the importance of wavelength, incidence angle, ice thickness and water depth, in determining transmission and reflection coefficients, and the pressure fluctuations on the bottom under the ice. It is felt that this model provides a good representation of the transmission of waves through large ice floes which, according to Robin,[7] have been observed to bend to allow waves to propagate through.

In Model III the ice is assumed to be made-up of floating point masses which are connected through a surface tension force. This model was, in fact, considered before the more complicated Model II, and many of the difficulties which arise in this more realistic case were first encountered and solved in the simpler boundary-value problem of Model III.

It is acknowledged that this surface tension model is not a realistic representation of the wave-ice interaction problem but it is of some academic interest since it extends the work of Keller and Goldstein[5] to finite depth of water. It is only given brief treatment since it follows closely the techniques used in Model II.

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PART I

MODEL 1. WAVE PROPAGATION THROUGH AN ICE-FIELD OF VARYING THICKNESS

I. INTRODUCTION AND EQUATIONS GOVERNING THE PROBLEM

A semi-infinite sheet of ice of variable thickness is at the surface of an ocean of uniform depth H ; the sheet occupies the domain $0 < y < \infty$, $-\infty < x < \infty$, $z = H$ and the remainder of the domain $z = H$ is open water. Ocean waves are assumed to impinge normally on the edge of the ice-sheet and the problem arises of the nature of the transmitted wave. Here we assume from Keller[5] that the problem can be approached using the approximations of shallow water theory, that is, we are assuming that the wavelength λ of the incident ocean waves is long compared with depth H of water. In the first place, if we approach the problem on the basis of the linearized theory only, the velocity potential $\Phi(x,y,z,t)$ for the liquid motion satisfies

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \quad , \; 0 < z < H \tag{1.1}$$

and

$$\Phi_z = 0 \quad , \quad z = 0 \tag{1.2}$$

At the free surface of the ocean where $\xi(x,y,t)$ is the elevation above the undisturbed level, we have, on the basis of linearized theory,

$$\frac{\partial \xi}{\partial t} = -\frac{\partial \Phi}{\partial z} \quad , \quad g\xi = \frac{\partial \Phi}{\partial t}$$

so that

$$\Phi_{tt} + g\,\Phi_z = 0 \quad , \quad z = H \; , \; -\infty < y < 0 \tag{1.3}$$

The difference of pressure across the ice-sheet is given by

$$p - p_o \simeq \rho\left(\frac{\partial\Phi}{\partial t} - g\xi\right)$$

and the equation of motion of an ice element, in the absence of bending stiffness and surface stress effects, is

$$p - p_o = \rho_i h(x,y)\xi_{tt}$$

where ρ_i is the ice-density and $h(x,y)$ the variable thickness of the ice. Hence, the condition to be satisfied on the ice-sheet will be

$$\begin{aligned} \frac{\partial\Phi}{\partial t} - g\xi &= s\ h(x,y)\xi_{tt} \\ \frac{\partial\xi}{\partial t} &= -\frac{\partial\Phi}{\partial z} \end{aligned} \tag{1.4}$$

where $s = \rho_i/\rho$ is the specific gravity of ice.

In order to simplify the problem, we now invoke the shallow water theory approximations and we do so by writing (see Keller[5])

$$\Phi(x,y,z,t) = \varphi(x,y,t) - \tfrac{1}{2}z^2(\varphi_{xx} + \varphi_{yy}) + 0(z)^3 \tag{1.5}$$

This expression will satisfy Eq. (1.1) to the first order and will also satisfy Eq. (1.2). In order to satisfy Eq. (1.3), the function $\varphi(x,y,t)$ must satisfy

$$\varphi_{tt} - g\ H(\varphi_{xx} + \varphi_{yy}) = 0 \quad , \quad -\infty < y < 0 \tag{1.6}$$

and, after some reduction of Eq. (1.4), we find that $\varphi(x,y,t)$ must satisfy

$$\varphi_{tt} - g\ H(\varphi_{xx} + \varphi_{yy}) - s\ Hh\ (x,y)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi_{tt} = 0,\ 0 < y < \infty \tag{1.7}$$

Equations (1.6) and (1.7) constitute the basic equations of the problem, the function $h(x,y)$ being the prescribed variable thickness of the ice.

We can take the function φ to contain the time through an exponential factor, and if we write

$$\varphi = F(x,y)\exp(-i\omega t) \tag{1.8}$$

then we have

$$F_{xx} + F_{yy} + \frac{\omega^2}{gH} F = 0 \quad , \; -\infty < y < 0 \tag{1.9}$$

and

$$\left\{1 - \frac{s\omega^2}{g} h(x,y)\right\} (F_{xx} + F_{yy}) + \frac{\omega^2}{gH} F = 0 \quad , \; 0 < y < \infty \tag{1.10}$$

At the boundary $y = 0$, it will be necessary to have φ and φ_y continuous, expressing the continuity of normal and transverse velocity, hence

$$F \;,\; F_y \quad \text{continuous at} \quad y = 0 \tag{1.11}$$

2. ICE THICKNESS VARYING IN y-DIRECTION WITH NORMALLY INCIDENT WAVES

Here we take the ice thickness h to vary only in the y-direction and, in order to simulate the effect of ice-floes, it will be assumed to have the following structure:

$$h = h_0 + h_1 \cos ky \tag{2.1}$$

where $h_1 \leq h_0$ and $2\pi/k$ is the wavelength of undulations in the ice.

We look for solutions in which the incident ocean waves impinge normally against the edge of the ice sheet, in which case the complete expression for φ in $-\infty < y < 0$ can be taken to be

$$\varphi = \exp\left\{i\omega\left(\frac{y}{c} - t\right)\right\} \tag{2.2}$$

where $c = \sqrt{gH}$. The transmitted wave in $0 < y < \infty$ will be of the form

$$\varphi = G(y)\ \exp(-i\omega t) \tag{2.3}$$

and from (1.10) it follows that $G(y)$ satisfies the differential equation

$$\left\{\left(1 - \frac{sh_o\omega^2}{g}\right) - \frac{sh_1\omega^2}{g}\cos ky\right\}\frac{d^2G}{dy^2} + \frac{\omega^2}{gH}G = 0 \tag{2.4}$$

It is not necessary to work with this differential equation in the above exact form since, when we insert typical value of the constants h_o , h_1 , s , ω , we find that

$$\frac{sh_1\omega^2}{g} \ll 1\ ,\ \frac{sh_o\omega^2}{g} \ll 1$$

for waves whose periodic time is greater than 4 seconds and we shall assume that the investigation is restricted to this class of waves. In this case it is permissible to expand

$$\left\{\left(1 - \frac{sh_o\omega^2}{g}\right) - \frac{sh_1\omega^2}{g}\cos ky\right\}^{-1}$$

in the form of a convergent infinite series in the parameter $sh_1\omega^2/(g-sh_o\omega^2)$ and thus to write (2.4) in the approximate form

$$\frac{d^2G}{dy^2} + \frac{\frac{\omega^2}{gH}}{\left(1 - \frac{sh_o\omega^2}{g}\right)}\left\{1 + \frac{\frac{sh_1\omega^2}{g}}{\left(1 - \frac{sh_o\omega^2}{g}\right)}\cos ky\right\} G = 0 \tag{2.5}$$

Accordingly, if we now write

$$ky = 2\eta \tag{2.6}$$

$$a = \frac{4\omega^2}{gHk^2} \frac{1}{\left(1 - \frac{sh_o\omega^2}{g}\right)} \tag{2.7}$$

$$q = \frac{2\omega^2}{gHk^2} \frac{\frac{sh_1\omega^2}{g}}{\left(1 - \frac{sh_o\omega^2}{g}\right)^2} \tag{2.8}$$

Equation (2.5) becomes

$$\frac{d^2G}{d\eta^2} + (a + 2q \cos 2\eta)G = 0 \tag{2.9}$$

which is the standard form of the Mathieu differential equation (McLachlan,[9] Theory and Application of Mathieu Functions). Periodic solutions of Mathieu's equation exist only under special circumstances and in order to make this clear we refer to the stability diagram in which q is the abscissa and a the ordinate (Fig. 1).

If the values of q and a are such that the representative point (q,a) in the stability diagram lies on the curves denoted by

$$a_o , a_1 , a_2 , \ldots$$
$$b_1 , b_2 , \ldots$$

there will be a periodic solution of Eq. (2.9). When q is sufficiently small, it is known (see McLachlan) that the equations of a_o , b_1 , a_1 , ... are as follows:

$$a_o = -\frac{1}{2}q^2 + \frac{7}{128}q^4 + 0(q^6) \tag{2.10}$$

$$b_1 = 1 - q - \frac{1}{8} q^2 + \frac{1}{64} q^3 - \frac{1}{1536} q^4 + 0(q^5) \qquad (2.11)$$

$$a_1 = 1 + q - \frac{1}{8} q^2 - \frac{1}{64} q^3 - \frac{1}{1536} q^4 + 0(q^5) \qquad (2.12)$$

$$b_2 = 4 - \frac{1}{12} a^2 + \frac{5}{13824} q^4 + 0(q^6) \qquad (2.13)$$

$$a_2 = 4 + \frac{5}{12} q^2 - \frac{763}{13824} q^4 + 0(q^6) \qquad (2.14)$$

and the solutions corresponding to a_0 , b_1 , a_1 , ... are as follows in the usual notation

$$a_0: \quad ce_0(\eta) = \sum_0^\infty A_{2n}^{(0)} \cos 2n\eta \qquad (2.15)$$

$$b_1: \quad se_1(\eta) = \sum_0^\infty B_{2n+1}^{(1)} \sin(2n+1)\eta \qquad (2.16)$$

$$a_1: \quad ce_1(\eta) = \sum_0^\infty A_{2n+1}^{(1)} \cos(2n+1)\eta \qquad (2.17)$$

Elsewhere in the stability diagram the solution for $G(\eta)$ of Eq. (2.9) is of the form

$$G(\eta) = e^{\eta\mu(\sigma)} \Phi(\eta,\sigma) + e^{-\eta\mu(\sigma)} \Phi(\eta,-\sigma) \qquad (2.18)$$

where $\Phi(\eta,\mu)$ is a periodic function of η , $\mu(\sigma)$ is the characteristic exponent of the Floquet-Poincaré theory, and σ is a parameter. We can illustrate the significance of the parameter σ by considering first of all, the area of the stability diagram lying between a_0 and b_2 .

Whittaker obtained the form of solution in this domain and he finds that (McLachlan, p. 70) one solution of (2.9) is given by $e^{\eta\mu(\sigma)}\ \Phi(\eta,\sigma)$ as follows:

$$\Phi(z,\sigma)= \sin(z-\sigma)+ s_3\sin(3z-\sigma)+ c_3\cos(3z-\sigma)+ s_5\sin(5z-\sigma)+\ldots \qquad (2.19)$$

and a and μ are expressed in terms of q and σ as follows:

$$a(\sigma,q) = 1-q\ \cos2\sigma+\frac{1}{4}q^2(-1+\frac{1}{2}\cos4\sigma)+\frac{1}{64}q^3\cos2\sigma + \ldots \qquad (2.20)$$

$$\mu(\sigma,q) = -\frac{1}{2}q\ \sin2\sigma + \frac{3}{128}q^3\sin2\sigma + \ldots \qquad (2.21)$$

It will be noted that $a(0,q) = b_1$, $a(-\frac{\pi}{2}, q) = a_1$, where b_1 , a_1 are the values given in (2.11) and (2.12). Whittaker finds that the value

$$\sigma = -\frac{1}{2}\pi + i\theta\ ,\ \theta \text{ real}\ ,\ \geq 0 \qquad (2.22)$$

gives the region between a_1 and b_2 and here μ is purely imaginary. This is therefore designated a stable area in the stability diagram.

In a similar way, the value

$$\sigma = i\theta\ ,\ \theta \text{ real}\ ,\ \geq 0 \qquad (2.23)$$

gives the stable region between a_0 and b_1 . The area between b_1 and a_1 is such that σ is real and lies in the range ,

$$-\frac{\pi}{2} \leq \sigma \leq 0 \qquad (2.24)$$

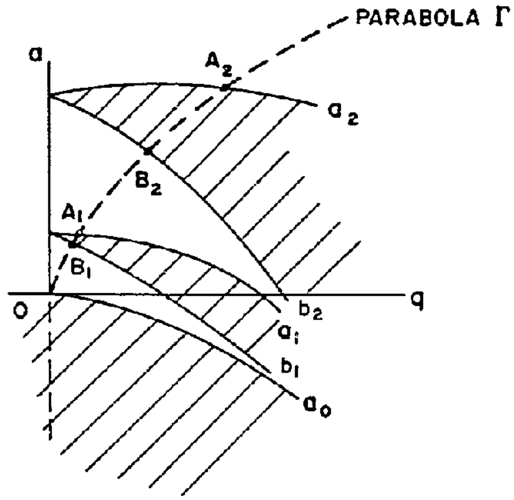
and this is designated an unstable area in the stability diagram. The remainder of the stability diagram can be investigated and described in a similar way.

The solution given in Eq. (2.18) indicates that the second solution can be deduced from the above described solution merely by a change in the sign of σ.

Returning now to the wave problem, we see that as long as the representative point (q,a) lies in any unstable domain (shown shaded in the stability diagram), the corresponding solution for $G(\eta)$ must be an attenuated transmitted wave; the degree of attenuation will depend upon the magnitude of $\mu(\sigma)$ and we see that in Eq. (2.21), μ will attain its maximum value in the unstable domain (b_1, a_1) for $\sigma = -\frac{\pi}{4}$ provided q is sufficiently small. On the other hand, if the representative point (q,a) lies on a_0, b_1, a_1, b_2, ... or within the stable unshaded areas (a_0, b_1), (a_1, b_2), ..., the solution for $G(\eta)$ will be an undamped transmitted wave. The character of this undamped transmitted wave will be periodic (but not a sine wave) when (q,a) actually lies on the curves a_0, b_1, a_1, b_2, ... ; the structure when (q,a) lies in a stable area such as (a_0, b_1) is not periodic but almost periodic. This description of regions of damped and undamped solutions can be expressed in a most useful form by returning to the definitions of a and q in Eqs. (2.7) and (2.8). It will be noted that each of these parameters contains the geometrical quantities H, h_0, h_1, the specific gravity s as well as the wavelength $2\pi/k$ of the ice undulations and the ocean wave frequency parameter ω; it will be observed also that the parameter ω, when eliminated between (2.7) and (2.8), leads to the relation

$$\omega^2 = \left(\frac{8}{sk^2h_1H}\right) q \tag{2.25}$$

For a fixed value of $(8/sk^2h_1H)$, this relation will be represented by a parabola Γ. This parabola will intersect the curves b_1, a_1, b_2, a_2, .. in points B_1, A_1, B_2, A_2, ... as shown in the diagram, and it is a straight-forward matter to locate these points analytically when q is not large. It is then possible to determine the values of ω at the positions B_1, A_1, B_2, A_2, ... which we can denote by $\omega(B_1)$, $\omega(A_1)$, ..



SKETCH 1

It then follows that the domains of ω defined by $0 < \omega < \omega(B_1)$, $\omega(A_1) < \omega < \omega(B_2)$, ... lead to stable or undamped transmitted waves, while the ranges

$$\omega(B_1) < \omega < \omega(A_1) \quad , \quad \omega(B_2) < \omega < \omega(A_2) \ , \ \ldots$$

give damped or attenuated transmitted waves. The breadth of these ranges will depend upon the parameter $(8/sk^2h_1H)$ which is seen therefore to play a crucial role. If we write

$$r = \frac{4}{sk^2h_1H} \tag{2.26}$$

the parabola Γ has the equation $a^2 = 2rq$ and the intersection of Γ with the curve b_1 , for example, can be obtained by combining (2.26) and (2.11), namely

$$a = 1 - q + 0(q^2) \tag{2.27}$$

so that the intersection is given by

$$2rq = (1 - q)^2$$

that is

$$q = (1 + r) - \sqrt{r(2 + r)}$$

and the value of ω then follows using (2.8). The approximate positions of $\omega(B_1)$, $\omega(A_1)$, ... can also be determined in this way.

To summarize the conclusions of this section of Part I, the important feature of the results is that when all the geometrical parameters are fixed, namely, the depth of water, average ice thickness and departure of ice thickness from the mean, the wavelength of undulations in the ice, there can be certain wavelength ranges for the incident ocean waves which will be attenuated on the ice-sheet and other wavelength ranges which will be transmitted as undamped waves.

3. MODIFICATION FOR OBLIQUELY INCIDENT WAVES

The above analysis requires modification whenever the incident wave impinges upon the ice-field at an angle other than zero. In order to carry out this modification, we refer to Eqs. (1.9) and (1.10). Clearly the solution of Eq. (1.9) representing an incident wave making an angle θ with the ice-field is

$$e^{iK_o y \cos\theta + iK_o x \sin\theta}$$

where

$$K_o = \frac{\omega}{\sqrt{gH}} = \frac{2\pi}{\lambda}$$

and λ is the wavelength of the incident wave.

Assume a solution of Eq. (1.10) of the form $H(x)\,G(y)$. Then substitution into (1.10) gives

$$\frac{H''(x)}{H(x)} = -\left\{\frac{G''(y)}{G(y)} + \frac{K_o^2}{1 - sHK_o^2 h(y)}\right\}$$

where the thickness of the ice is assumed to vary in the y-direction only. Since the left-hand side is a function of x only, while the right-hand side is a function of y only, each side must be equal to a constant. To obtain an oscillatory x-variation for all x, $-\infty < x < \infty$, it is necessary that this constant be negative. Further, in order that the solutions for $y < 0$ and $y > 0$ might be continuous at $y = 0$ for all x, $-\infty < x < \infty$, it is necessary for the x-dependence to be the same for $y > 0$ as for $y < 0$. Thus

$$\frac{H''(x)}{H(x)} = - K_o^2 \sin^2 \theta$$

and

$$H(x) = e^{iK_o x \sin \theta}$$

Thus the equation satisfied by $G(y)$ is

$$\frac{d^2G}{dy^2} + K_o^2\left[\frac{1}{1 - sHK_o^2h(y)} - \sin^2\theta\right] G(y) = 0 \qquad (3.1)$$

Substituting the assumed thickness distribution $h(y) = h_o + h_1 \cos ky$ into Eq. (3.1) and making the same approximations as before, namely

$$sh_1 K_o^2 H << 1 \text{ , } sh_o K_o^2 H << 1$$

gives

$$\frac{d^2G}{d\eta^2} + (a + 2q \cos 2\eta)\ G(\eta) = 0$$

where in this case

$$a = \frac{4K_o^2}{k^2} \left\{ \frac{1}{1 - shK_o^2\ h_o} - \sin^2 \theta \right\} \qquad (3.2)$$

$$q = \frac{2K_o^2}{k^2} \frac{sh_1 K_o^2 H}{(1 - sh_o K_o^2 H)^2} \tag{3.3}$$

and the substitution $ky = 2\eta$ has been made. Once again Mathieu's differential equation is obtained. Here also the frequency dependence may be eliminated to obtain a relation between a and q, but since, in this case, the relation is no longer simple, there seems to be no advantage in doing this.

4. DISCUSSION OF RESULTS

(a) Existence of Energy Cut-Off And Its Dependence on the Various Parameters

The present mathematical model, based on the shallow-water approximation, with ice undulations following a cosine distribution, clearly does not represent an actual ice-field. However, a number of qualitative results may be deduced which might be expected to hold true for more realistic distributions of ice floes. Thus it has been shown that for fixed values of k, h_1, H, there may or may not be undamped transmission of a given incident wave into the ice-field. By undamped transmission is meant a wave travelling through the ice undiminished in amplitude with distance into the ice. An attenuated transmitted wave is a wave which decays exponentially and travels only a short distance into the ice. Thus for $\theta = 0$, whenever the parabola given by Eq. (2.25) crosses a shaded region in Fig. 1, the transmitted wave is attenuated, whereas when the parabola crosses an unshaded region, the wave will be unattenuated and will proceed through the ice. Thus the domains of ω defined by

$$0 < \omega < \omega(B_1) \ , \ \omega(A_1) < \omega < \omega(B_2), \ \ldots$$

denote stable or undamped transmitted waves, while the ranges

$$\omega(B_1) < \omega < \omega(A_1) \ , \ \omega(B_2) < \omega < \omega(A_2), \ \ldots$$

denote damped or attenuated waves.

Now for shallow water, the relation between wavelength λ and frequency ω is given by

$$\lambda = \frac{2\pi\sqrt{gH}}{\omega}$$

so that corresponding to the domains of ω described above we have the wave-bands

$$\lambda > \lambda(B_1)\ ,\ \lambda(A_1) > \lambda > \lambda(B_2),\ \ldots$$

denoting wavelengths of incident waves which are transmitted through the ice as undamped waves, and the wave-bands

$$\lambda(B_1) > \lambda > \lambda(A_1)\ ,\ \lambda(B_2) > \lambda > \lambda(A_2),\ \ldots$$

corresponding to attenuated transmitted waves.

One striking observation which may be made is that there exists a wavelength

$$\lambda(B_1) = \frac{2\pi\sqrt{gH}}{\omega(B_1)}$$

such that all incident waves whose wavelength satisfies $\lambda > \lambda(B_1)$ are transmitted through the ice. When $\lambda = \lambda(B_1)$ the first energy cut-off occurs and complete reflection of the incident wave takes place for all λ, satisfying $\lambda(A_1) < \lambda < \lambda(B_1)$. For $\lambda(B_2) < \lambda < \lambda(A_1)$ wave transmission again occurs. The distribution and width of the stable and unstable regions are governed by the various parameters in the problem, and it is possible to make the following general observations.

As $h_1 \to 0$, corresponding to a uniform constant thickness h_o, then $q \to 0$ and Mathieu's equation degenerates into the equation

$$\frac{d^2G}{d\eta^2} + aG(\eta) = 0$$

whose solution does not of course exhibit stable and unstable regions. This may also be seen by considering the parabola

$$a^2 = \frac{8}{sk^2 h_1 H} q \qquad (\theta = 0) \tag{3.4}$$

For small h_1 the parabola becomes steeper and the width of the unstable regions becomes smaller, there being no unstable regions in the limit $h_1 = 0$ (see Sketch 1, p. 13).

A similar argument can be made for the case of small k , corresponding to long wave undulations in the ice, and small H , corresponding to very shallow water. In each case the parabola (Eq. [3.4]) is very steep and the width of the unstable regions are small, vanishing altogether in the limit of H , $k \equiv 0$. This also follows from a consideration of the limit of the Eq. (2.5) for small H and k .

It appears from Eq. (3.2) that the effect of finite incidence angle θ is to reduce the value of a , while q remains constant, thus moving points on the stability diagram into regions of greater instability. For angles close to 90°, where a is close to zero, but still positive, Sketch 1 indicates that the first stable region $0 < \omega < \omega(B_1)$ is wide, but that subsequent regions are predominantly unstable. Thus the wavelength $\lambda(B_1)$ at which the first energy cut-off occurs is small, and only narrow bands of wavelengths smaller than $\lambda(B_1)$ penetrate the ice.

Further information concerning the effect of the various physical parameters upon wave reflection and transmission was obtained by making some computations which are described in the following section.

(b) Numerical Results

For the case of normally incident waves ($\theta = 0^\circ$), some computations were made to determine the breadth of the unstable regions corresponding to complete reflection of the incident wave. Dimensions corresponding to model sizes were chosen which were then scaled using scale ratios of 80:1 and 200:1. Thus, ice thicknesses of $h_o = h_1 = 0.25$, 1.00 and 2.00 inches were considered, with wavelengths of undulations in the ice (i.e., floe length) of 0.50, 1.00, and 3.00 feet. The water depth was taken to be 0.50 feet and the specific gravity of ice $s = 0.92$. Then from Eq. (2.25) the quantity a was determined for particular values of q. The equations of the curves a_0, b_1, a_1, b_2 up to b_6 were computed using the tables in Appendix II of McLachlan[9] for $q > 1$, together with the series expansions given by Eqs. (2.10) to (2.14) for $q < 1$ (see McLachlan,[9] p. 16-17). These curves were then drawn and the intersection of the parabola

$$a^2 = \frac{8}{sk^2 h_1 H} q$$

with each curve a_0, b_1, ... b_6, tabulated. The intersection points give values of q from which the frequency and hence the wavelength of the incident wave can be determined using Eq. (2.8). These points which span shaded regions (Fig. 1) denoting unstable solutions of Mathieu's equation, determine bands of wavelengths corresponding to incident waves which are completely reflected by the ice-field. The results are given in Table I. All wavelengths which lie outside the ranges indicated in the table correspond to incident waves which are transmitted through the ice as undamped waves. The largest wavelength given in each case is $\lambda(B_1)$, and all waves having a wavelength $\lambda > \lambda(B_1)$ propagate into the ice as undamped waves.

From Fig. 1, as q and hence, ω, increases indefinitely, the width of the stable regions intersecting the parabola (2.25) diminishes until a value of q is reached above which the bands of stable frequencies are indistinguishable from points. Since the stability curves all cross the real axis for large enough q (see McLachlan,[9] p. 39), the number of intersection "points" is infinite. Associated with this value of q is a wavelength which we denote by λ_{min}. Thus, we define λ_{min}, as the wavelength below which the bands of stable wavelengths intersected by the parabola (Eq. [2.25]), are sensibly points, corresponding to very little penetration of the ice. Clearly this is somewhat arbitrary, depending as it does on the accuracy of computation, but λ_{min} does provide us with a useful bound below which only discrete wavelengths penetrate the ice-field.

Similarly, for small q , and hence ω , the width of the unstable regions intersected by the parabola (Eq. [2.25]) diminishes and reduces to "points." Also for large floes (k small), or thin ice (h_o, h_1 small) the parabola is steeper and the bands of instability given by the intersection are diminished.

In the table, both stable and unstable bands which are sensibly points are omitted. As an illustration of the use of the table, consider the case of water of depth 100 feet, and ice of thickness 4.17 feet. Then for floe lengths of 100 feet, $\lambda_{min} = 162$ there being no wave bands (but an infinity of wave "points") which penetrate the ice for $\lambda < 162$, whereas if the ice thickness is 16.67 feet, λ_{min} has more than doubled to 344. Also, note that all wavelengths penetrate a floe 600 feet long and 4.17 feet thick (apart from an infinity of discrete wavelengths) but if the floe is 16.67 feet thick, there exists bands of wavelengths which fail to penetrate the ice.

(c) Comparison of Results with Observation

A unique study of waves in pack ice was made by Robin[7,8] during a voyage into the Weddell Sea aboard RRS JOHN BISCOE in 1959-60. He finds that for floes of around 1.5m thick and 40m or less in diameter,

> "...the main energy cut-off took place when floe diameters were about one-third of the wavelength; little loss of energy occurred when floes were less than one-sixth of the wavelength across, while no detectable penetration took place when the floes were half a wavelength or more in diameter."

From the table, the closest comparison can be made for floes of 4.17-ft thick and 100-ft long. Then little penetration takes place for $\lambda < 162$ corresponding to a floe length of five-eights the wavelength or more as compared to the half wavelength observed by Robin.[7,8] It is not possible to estimate the energy loss for long waves as this requires knowledge of the transmitted wave amplitude and hence the full solution of Mathieu's equation. However the table indicates that the first unstable region corresponding to an energy cut-off or complete reflection of the incident wave occurs when λ drops to 248 feet or when the floe length is about two-fifths of the wavelength. This compares favorably with the observed value of one third of a wavelength given by Robin.[7,8]

5. CONCLUSIONS

It would appear that the simplified model considered here confirms qualitatively the following observations of Robin:

(1) The existence of a critical wavelength at which a major energy cut-off occurs.

(2) The existence of a wavelength below which very little penetration of the ice-field takes place.

The one numerical comparison made shows surprisingly good agreement of theory with observation. It would seem that a study of more realistic thickness distributions would prove fruitful.

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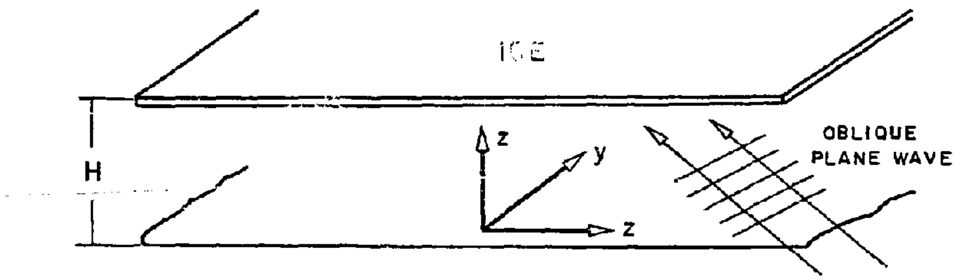
PART II

A. MODEL II. WAVE TRANSMISSION THROUGH A FLEXIBLE ICE-FIELD

1. FORMULATION



SKETCH 2

A semi-infinite ice sheet is floating on the surface of water of constant depth H. A plane wave is obliquely incident from the region $-\infty < y < 0$. Let $\Phi(x,y,z,t)$ be the velocity potential of the liquid motion. Then on the linearized theory of small amplitude water waves, Φ satisfies

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \qquad 0 < z < H \tag{1.1}$$

$$\Phi_z = 0 \quad , \quad z = 0 \tag{1.2}$$

$$\Phi_{tt} + g\Phi_z = 0 \quad \text{at} \quad z = H \tag{1.3}$$

$$-\infty < y < 0 \text{ (free surface)}$$

Let the equation of the ice field be

$$z = \xi(x,y,t) + H \tag{1.4}$$

where $\xi(x,y,t)$ represents the displacements of the ice sheet above the undisturbed free surface $z = H$. Then Bernoulli's equation gives

$$p - p_o = \rho\left(\frac{\partial\Phi}{\partial t} - g\xi\right) \tag{1.5}$$

and we also have

$$\frac{\partial\Phi}{\partial z} = -\frac{\partial\xi}{\partial t} \tag{1.6}$$

on $z = H$, on the linear theory.

It is assumed that the ice sheet, having constant mass thickness h, is displaced from equilibrium by the differential pressure $p - p_o$, and that each element will be subjected to a force arising from the bending stresses in the sheet.

It may be shown[10] that

$$p - p_o = D\nabla^4 \xi + \rho_i h(\xi_{tt} + g) \tag{1.7}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad \begin{aligned} E &= \text{Young's modulus} \\ \nu &= \text{Poisson's ratio} \\ \rho_i &= \text{density of ice} \end{aligned}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}$$

Combining (1.5) and (1.7) and differentiating with respect to t gives

$$\Phi_{tt} + g\Phi_z = -\frac{D}{\rho}\nabla^4 \Phi_z - \frac{\rho_i h}{\rho}\Phi_{ttz} \quad , \quad z = H \quad , \quad 0 < y < \infty \tag{1.8}$$

where (1.6) has been used.

Since the ice extends to infinity in either x-direction, it is possible to subtract out the x-dependence. Assuming also a time harmonic dependence of frequency ω , let

$$\Phi(x,y,z,t) = \mathrm{Re}\left\{ \phi(y,z)\, e^{ikx} \quad e^{-i\omega t} \right\} \tag{1.9}$$

whence $\phi(y,z)$ satisfies

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - k^2\phi = 0 \quad , \quad 0 < z < H \quad , \quad -\infty < y < \infty \tag{1.10}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad , \quad z = 0 \quad , \quad -\infty < y < \infty \tag{1.11}$$

$$K\phi = \frac{\partial \phi}{\partial z} \quad , \quad z = H \quad , \quad -\infty < y < 0 \tag{1.12}$$

$$K\phi = (1-L)\frac{\partial \phi}{\partial z} + M\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 \frac{\partial \phi}{\partial z} \quad , \quad z = H \quad , \quad 0 < y < \infty \tag{1.13}$$

where $K = \omega^2/g = \frac{2\pi}{\lambda}$, $M = D/\rho g$, $L = \frac{\rho_i K h}{\rho}$

Conditions (1.10) to (1.13) are not by themselves sufficient to give a unique solution $\phi(y,z)$. Additional conditions regarding the vanishing of the bending moment and shearing force at the edge of the ice field will be imposed together with assumptions regarding the form of the solution for $y = \pm\infty$. Assumptions concerning the behavior of ϕ and its derivatives near $y = 0$, $z = H$ which ensure that Fourier transforms converge will be made during the course of the analysis. These assumptions may be verified once the final solution is obtained.

Note that the particular case $k = 0$ corresponds to a plane wave which is normally incident upon the ice field; that is, whose crests are parallel to $y = 0$.

2. PRELIMINARY DISCUSSION OF THE SOLUTION

The eigenfunctions of Eqs. (1.10), (1.11) and (1.12) for $y < 0$ are $\exp\ [\pm y\ (k^2 + a_n^{\ 2})^{\frac{1}{2}}]\ \cos\ a_n z (n = 1,2, \ldots)$ where the a_n are the roots of the equation,

$$K \cos a_n H + a_n \sin a_n H = 0 \tag{2.1}$$

It is shown in Appendix A that there are an infinite number of real roots a_n such that $|a_n| < |a_{n+1}|$, $(n = \pm 1, \pm 2, \ldots)$ and that for $K > 0$ there are also two pure imaginary roots $\pm\ ia_o$ with eigenfunctions $\exp\ [\pm y\,(k^2 - a_o^{\ 2})^{\frac{1}{2}}]\ \cosh\ a_o z$. There are no other roots. For a plane wave, $k^2 < a_o^{\ 2}$ so that there may be propagation in either y-direction for $y < 0$, since the exponent is purely imaginary. Thus for y negative, the expected form of a bounded solution would be

$$\exp\ [\pm\ iy\,(a_o^{\ 2} - k^2)^{\frac{1}{2}}]\ \cosh\ a_o z + 0\ (\exp ky)$$

For $y > 0$, the situation is more complicated. The eigenfunctions of Eqs. (1.10), (1.11) and (1.13) are of the form

$$\exp\ [\pm y\ (k^2 + b_n^{\ 2})^{\frac{1}{2}}]\ \cos\ b_n z\ \ (n = 1,2, \ldots)$$

where the b_n are the roots of the equation

$$K \cos b_n H + b_n \left(1 - L + Mb_n^{\ 4}\right) \sin b_n H = 0 \tag{2.2}$$

A detailed examination of the roots of this equation is made in Appendix A. It is found that for $L < 1$, $M > 0$, Eq. (2.2) has two purely imaginary roots $\pm\ ib_o$, a doubly infinite sequence of real roots $\pm\ b_n (n = 1,2, \ldots)$, and four complex roots $\pm\ c_o$, $\pm\ \bar{c}_o$ (the assumption $L < 1$ covers the range of practical interest; the case $L > 1$ is not considered here). The exact position of the complex roots is not required. It is sufficient for the time being to note that $c_o = \lambda + i\mu$ where $\lambda > \mu > 0$; this is proved in Appendix A. Because of this, c_o^2 lies in the first quadrant, and $\mathrm{Re}\ c_o^2 > 0$. Then it follows that $\mathrm{Re}\ c_o' > k$ where $c_o' = (c_o^2 + k^2)^{\frac{1}{2}}$ the square root being such that $c_o' = c_o$ when $k = 0$. A similar inequality holds for $\bar{c}_o' = (\bar{c}_o^2 + k^2)^{\frac{1}{2}}$. Thus for $y > 0$ there exist eigenfunctions of the form

$$\exp\left\{\pm y\ (k^2 - b_o^2)^{\frac{1}{2}}\right\} \cosh\ b_o z$$

which provide the wave propagation if $b_o^2 > k^2$,

$$\exp\left\{\pm y\ (k^2 + b_n^2)^{\frac{1}{2}}\right\} \cosh\ b_o z\ ,\ (n = 1,2,\ \ldots)\ ,$$

$$\exp\left\{\pm y\ c_o'\right\} \cosh\ c_o z$$

and

$$\exp\left\{\pm y\ \bar{c}_o'\right\} \cosh\ \bar{c}_o z$$

Thus for y positive, the expected form of a bounded solution would be

$$\exp\left\{\mp iy\ (b_o^2 - k^2)^{\frac{1}{2}}\right\}\ \cosh\ b_o z + O(e^{-ky})$$

if $b_o^2 > k^2$, the terms $O(e^{-ky})$ arising from the fact that $\mathrm{Re}\ c_o' = \mathrm{Re}\ \bar{c}_o' > k$. If $b_o^2 < k^2$, then there will be no wave propagation into the ice. This is made clear in Fig. 2. For the incident wave, we write $k = a_o \sin\theta$, and for the transmitted wave, $k = b_o \sin\theta_T$ where θ_T is the angle of transmission.

Then the incident wave is of the form

$$e^{ia_o y \cos\theta + ia_o x \sin\theta} \cosh\ a_o z$$

whereas the transmitted wave takes the form

$$e^{ib_o y \cos\theta_T + ib_o x \sin\theta_T} \cosh\ b_o z$$

Clearly, since

$$\sin\theta_T = \frac{k}{b_o} = \frac{a_o \sin\theta}{b_o} \tag{2.3}$$

propagation into the ice will only occur if $a_o \sin\theta < b_o$. Whenever $a_o \sin\theta > b_o$, the transmitted wave becomes exponentially damped of the form

$$e^{-y(a_o^2 \sin^2\theta - b_o^2)^{\frac{1}{2}}} \cosh\ b_o z$$

so that in this case <u>all</u> modes decay exponentially and the incident wave is totally reflected. Thus for given K, L, M, there exists a critical incident angle θ_{crit} such that an incident wave approaching the ice-field at an angle $\theta > \theta_{crit}$ is completely reflected (see Fig. 3). This will occur when $\theta_T = 90^\circ$ so that

$$\theta_{crit} = \sin^{-1}(b_o/a_o) \tag{2.4}$$

The transmitted angle θ_T indicates whether a given incident wave will be bent towards or away from the normal to the ice field. If $b_o < a_o$ then from Eq. (2.3), $\theta < \theta_T$, the transmitted wave is bent away from the normal, and there will always exist a critical angle given by Eq. (2.4). On the other hand, if $a_o < b_o$, then $\theta_T < \theta$ and the transmitted wave is bent towards the normal (see Fig. 4). In this case, an incident wave approaching the ice-field will always penetrate the ice regardless of the incident angle. If $a_o = b_o$, no deviation of the incident wave occurs.

For the time being, it will be assumed that $a_o \sin\theta < b_o$ so that there exists an undamped progressive wave travelling into the ice-field. In the case of normal incidence, $\theta = 0$, there will always exist such a wave.

3. METHOD OF SOLUTION

(a) Derivation of the Wiener-Hopf Equation

The solution for the function $\phi(y,z)$ is achieved by means of Fourier transforms and the Wiener-Hopf technique. In order that the Fourier transforms might converge in a strip of the transform variable plane, the following device is used. The preceding section indicates the expected form of the solution for large values of y. It is anticipated that a prescribed oblique plane wave incident from $y < 0$ will give rise to a reflected wave in $y < 0$, and a transmitted wave in $y > 0$. The amplitude and phase of the reflected and transmitted waves are determined once the incident wave is prescribed. However, we shall fix the amplitude and phase of the transmitted wave beforehand and, hence, determine the reflected and incident waves. The reason for this will soon become apparent.

Thus, let

$$\phi(y,z) = e^{ib_o'y} \cosh b_o z + O(e^{-ky}) \quad , \quad 0 \leq z \leq H \quad , \; y > 0 \tag{3.1}$$

where $b_o' = (b_o^2 - k^2)^{\frac{1}{2}}$ and $b_o' = b_o$ when $k = 0$

and
$$\phi(y,z) = Ie^{+ia_o'y} \cosh a_o z + Re^{-ia_o'y} \cosh a_o z + O(e^{ky}) \; , \quad 0 \leq z \leq H \; , \; y < 0 \tag{3.2}$$

where

$$a_o' = (a_o^2 - k^2)^{\frac{1}{2}} \quad \text{and} \quad a_o' = a_o \quad \text{when} \quad k = 0$$

Division of the solution by I gives the solution due to an incident potential $e^{ia_o'y} \cosh a_o z$.

Consider, now, the function $\psi(y,z)$ where

$$\psi(y,z) = \phi(y,z) - e^{ib_o'y} \cosh b_o z$$

Then from (3.1) $\psi(y,z)$ is exponentially small for $y > 0$ so that the Fourier transform of $\psi(y,z)$ with respect to y will exist in a strip of the transform plane; a basic requirement for the successful application of the Wiener-Hopf technique.

Now, $\psi(y,z)$ satisfies

$$\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - k^2\psi = 0 \ , \ 0 < z < H \ , \ -\infty < y < \infty \qquad (3.3)$$

$$\frac{\partial\psi}{\partial z} = 0 \ , \ z = 0 \ , \ -\infty < y < \infty \qquad (3.4)$$

$$K\psi = \frac{\partial\psi}{\partial z} + Ae^{ib_o'y} \ , \ z = H \ , \ -\infty < y < 0$$

where $A \equiv (Mb_o^4 - L)b_o \sinh b_o H$

$$K\psi = (1-L)\frac{\partial\psi}{\partial z} + M\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2\frac{\partial\psi}{\partial z} \ , \ z = H \ , \ 0 < y < \infty \qquad (3.6)$$

$$\psi(y,z) = O(e^{-ky}) \ , \ y > 0 \ , \ \text{for each } z \qquad (3.7)$$

$$\psi(y,z) = Ie^{+ia_o'y}\cosh a_o z + Re^{-ia_o'y}\cosh a_o z - e^{ib_o'y}\cosh b_o z + O(e^{ky}),$$

$$y < 0 \ \text{ for each } z \qquad (3.8)$$

It is assumed that outside some neighborhood of $(0,H)$ ψ and its first and second partial derivatives are also

$$O(e^{-ky}) \ , \ y > 0 \quad \text{and} \quad O(1) \ , \ y < 0 \qquad (3.9)$$

It is further assumed that ψ is bounded everywhere and that in a neighborhood of (0,H)

$$\frac{\partial\psi}{\partial y} \;,\; \frac{\partial\psi}{\partial z} \;=\; 0(\frac{1}{r^{\beta}}) \quad , \quad 0 < \beta < 1 \tag{3.10}$$

where $r^2 = y^2 + (z - H)^2$

The reason for the assumption given by Eq. (3.10) requires some explanation. There is no a priori reason why the velocity components should be non-singular at the edge of the ice-field. Such singularities invariably occur in potential problems at the confluence of two boundaries on which different boundary conditions are satisfied. However, physically, we require that there be no breaking of waves at the interface as this would introduce an arbitrary constant to determine the amount of energy loss which occurs. On a linear theory such a breaking phenomenon is represented by a sink singularity in ψ so that the velocity components would be $0(\frac{1}{r})$ in the neighborhood of the leading edge, corresponding to the logarithmic singularity in ψ. By insisting that the singularity be of the form (3.10), loss of energy due to breaking is just avoided, at the same time allowing any milder singularity to occur in the analysis.

From conditions (3.9) and (3.10), the Fourier transform

$$\Psi(\alpha,z) = \int_{-\infty}^{\infty} \psi(y,z)e^{i\alpha y}dy \tag{3.11}$$

exists for $0 \le z \le H$, and is regular for α $(= \sigma + i\tau)$ in the strip D: $-k < \tau < 0$ of the complex α-plane (see Sketch 3, p. 37). From Eq. (3.3)

$$\int_{-\infty}^{\infty}(\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} - k^2\psi)e^{i\alpha y}dy = 0$$

and integration by parts gives

$$\left[\frac{\partial^2}{\partial z^2} - (\alpha^2 + k^2)\right] \Psi(\alpha,z) = 0 \quad , (\alpha \in D \;,\; 0 \le z < H)$$

Define $\gamma = (\alpha^2 + k^2)^{\frac{1}{2}}$ such that the cuts extend from $\pm ik$ to $\pm i\infty$ along the imaginary axis in the α-plane. Then,

(i) As $\alpha = \sigma$ (real) $\rightarrow \infty$, $\gamma(\sigma) \sim \sigma$

(ii) Re $(\gamma) > 0$ everywhere away from the cuts

and

$$\Psi(\alpha,z) = A_1(\alpha) \cosh \gamma z + A_2(\alpha) \sinh \gamma z$$

The transform of condition (3.4) is

$$\lim_{z \to 0} \frac{\partial \Psi}{\partial z}(\alpha,z) = 0$$

so that $A_2(\alpha) \equiv 0$.

From condition (3.10) it may be shown that

$$\lim_{z \to H-} \Psi(\alpha,z) = \Psi(\alpha,H)$$

and

$$\lim_{z \to H-} \frac{\partial \Psi}{\partial z}(\alpha,z) = \frac{\partial \Psi}{\partial z}(\alpha,H)$$

Thus

$$\Psi(\alpha,z) = \Psi(\alpha,H) \frac{\cosh \gamma z}{\cosh \gamma H} \qquad (\alpha \in D) \tag{3.12}$$

and $$\frac{\partial \Psi}{\partial z}(\alpha,H) = \gamma H \tanh \gamma H \ \Psi(\alpha,H) \qquad (\alpha \in D) \tag{3.13}$$

In order to obtain a Wiener-Hopf equation, the following notation is used. Let

$$\Psi(\alpha,z) = \Psi_+(\alpha,z) + \Psi_-(\alpha,z)$$

where

$$\Psi_+(\alpha,z) = \int_0^\infty \psi(y,z)e^{i\alpha y}dy \tag{3.14}$$

exists for $0 \le z \le H$ and is a regular function of α in D_+: $\tau > -k$ and

$$\Psi_-(\alpha,z) = \int_{-\infty}^0 \psi(y,z)e^{i\alpha y}dy \tag{3.15}$$

exists for $0 \le z \le H$ and is a regular function of α in D_-: $\tau < 0$

Then

$$\lim_{z \to H-} \Psi_\pm(\alpha,z) = \Psi_\pm(\alpha,H)$$

$$\lim_{z \to H-} \frac{\partial\Psi_\pm}{\partial z}(\alpha,z) = \frac{\partial\Psi_\pm}{\partial z}(\alpha,H)$$

In future $\Psi_\pm(\alpha,H)$ and $\frac{\partial\Psi_\pm}{\partial z}(\alpha,H)$ will be abbreviated to $\Psi_\pm$ and $\Psi_\pm'$, respectively.

Now the transform of condition (3.5) is easily seen to be

$$K\Psi_- = \Psi_-' - \frac{iA}{\alpha + b_0'} \quad (\alpha \in D_-) \tag{3.16}$$

Condition (3.6) is more troublesome. It is assumed that all y derivatives of $\frac{\partial\psi}{\partial z}(y,H)$ up to and including $\frac{\partial^5\psi}{\partial y^4\partial z}(y,H)$ are

$$O(e^{-ky}) \quad , \quad y > 0 \tag{3.17}$$

It is also assumed that

$$\frac{\partial^5\psi}{\partial y^4\partial z}(y,H) \quad \text{is} \quad O\left(\frac{1}{y^\beta}\right) \qquad (0 < \beta < 1) \text{ as } y \to 0+ \tag{3.18}$$

Then the transform of condition (3.6) exists and integration by parts gives

$$\int_0^\infty \left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 \frac{\partial \psi}{\partial z}(y,H)\, e^{i\alpha y}\, dy = - c_4 + i\alpha c_3 + (\alpha^2 + 2k^2)(c_2 - i\alpha c_1) + \gamma^4 \, \Psi_+{}' \quad (\alpha \in D_+) \tag{3.19}$$

where condition (3.17) has been used, and where

$$c_1 = \left[\frac{\partial \psi}{\partial z}(y+,H)\right]_{y=0+} \qquad c_2 = \frac{\partial}{\partial y}\left[\frac{\partial \psi}{\partial z}(y,H)\right]_{y=0+}$$

$$c_3 = \frac{\partial^2}{\partial y^2}\left[\frac{\partial \psi}{\partial z}(y,H)\right]_{y=0+} \qquad c_4 = \frac{\partial^3}{\partial y^3}\left[\frac{\partial \psi^3}{\partial z}(y,H)\right]_{y=0+}$$

Thus the transform of condition (3.16) becomes

$$K\Psi_+ = \Psi_+{}' \,[1 - L + M\gamma^4] - M\,[c_4 - i\alpha c_3 - (\alpha^2 + 2k^2)(c_2 - i\alpha c_1)] \quad (\alpha \in D_+) \tag{3.20}$$

Now add Eq. (3.16) to Eq. (3.20) and use Eq. (3.13). Then

$$\left\{K \cosh \gamma H - \gamma\,(1 - L + M\gamma^4) \sinh \gamma H\right\} \Psi_+{}' + \left\{K \cosh \gamma H - \gamma \sinh \gamma H\right\} \Psi_-{}'$$
$$= - \gamma \sinh \gamma H \left\{\frac{iA}{\alpha + b_o{}'} + M\left[c_4 - i\alpha c_3 - (\alpha^2 + 2k^2)(c_2 - i\alpha c_1)\right]\right\} \quad (\alpha \in D) \tag{3.21}$$

Equation (3.21) is a typical Wiener-Hopf equation holding in a strip of the complex α-plane. Although Eq. (3.21) involves the constants $c_i (i=1,2,3,4)$, it will be shown how Ψ' and hence Ψ may be determined in terms of just two constants.

Let

$$\begin{aligned} K \cosh \gamma H - \gamma(1 - L + M\gamma^4) \sinh \gamma H &= f_1(\alpha) \\ K \cosh \gamma H - \gamma \sinh \gamma H &= f_o(\alpha) \end{aligned} \tag{3.22}$$

Then

$$f_1 - f_o = -(M\gamma^4 - L)\ \gamma \sinh \gamma H \tag{3.23}$$

and Eq. (3.21) becomes

$$f_1 \left\{ (M\gamma^4 - L)\ \Psi_+' - \frac{iA}{\alpha + b_o'} - M\ [c_4 - i\alpha c_3 - (\alpha^2 + 2k^2)(c_2 - i\alpha c_1)] \right\}$$

$$= - f_o \left\{ (M\gamma_4 - L)\ \Psi_-' + \frac{iA}{\alpha + b_o'} + M\ [c_4 - i\alpha c_3 - (\alpha^2 + 2k^2)(c_2 - i\alpha c_1] \right\} \quad (\alpha \in D) \tag{3.24}$$

It is shown in Appendix B that we may define

$$\frac{f_1(\alpha)}{f_o(\alpha)} \equiv \frac{K_+(\alpha)}{K_-(\alpha)} \tag{3.25}$$

where $K_\pm(\alpha)$ is regular and non-zero in $D_\pm$, respectively. Furthermore, it is shown that

$$K_+(\alpha) = 0(\alpha^2) \quad , \quad |\alpha| \to \infty \ , \ (\alpha \in D_+) \tag{3.26}$$

and

$$K_-(\alpha) = 0(\frac{1}{\alpha^2}) \quad , \quad |\alpha| \to \infty \ , \ (\alpha \in D_-) \tag{3.27}$$

Then (3.24) may be written

$$K_+(\alpha) \left\{ (M\gamma^4 - L)\ \Psi_+' - M\ [c_4 - i\alpha c_3 - (\alpha^2 + 2k^2)(c_2 - i\alpha c_1)] \right\} - iA \left[\frac{K_+(\alpha) - K_+(-b_o')}{\alpha + b_o'} \right]$$

$$= - K_-(\alpha) \left\{ (M\gamma^4 - L)\ \Psi_-' + M\ [c_4 - i\alpha c_3 - (\alpha^2 + 2k^2)(c_2 - i\alpha c_1] \right\}$$

$$- iA \left[\frac{K_-(\alpha) - K_+(-b_o')}{\alpha + b_o'} \right] \ , \ (\alpha \in D) \tag{3.28}$$

It is clear that each term on the left-hand side of (3.28) is regular for $\alpha \in D_+$, whereas each term on the right-hand side is regular for $\alpha \in D_-$. Since the two sides are equal for $\alpha \in D$, this defines a function $J(\alpha)$ regular for all (finite) α. The determination of $J(\alpha)$ is achieved by a consideration of its behavior as $|\alpha| \to \infty$.

Consider the left-hand side of Eq. (3.28). From (3.20), it is equal to

$$K_+(\alpha)\left\{K\Psi_+ - \Psi_+'\right\} - iA\left\{\frac{K_+(\alpha) - K_+(-b_o')}{\alpha + b_o'}\right\} \quad , \qquad (\alpha \in D_+)$$

Now Ψ_+, $\Psi_+' \to 0$ as $|\alpha| \to \infty$ in D_+ from condition (3.10), (see Noble,[11] p. 36, Eq. [1.74]).*

Thus, since $K_+(\alpha) = O(\alpha^3)$, $|\alpha| \to \infty$ in D_+, then $J(\alpha) = o(\alpha^2)$ as $|\alpha| \to \infty$ in D_+. Similarly, since $K_-(\alpha) = O(\frac{1}{\alpha^2})$ as $|\alpha| \to \infty$ in D_-, and $\Psi_-' \to 0$ as $|\alpha|$ in D_-, then $J(\alpha) = o(\alpha^2)$ as $|\alpha| \to \infty$ in D_-.

Hence

$$J(\alpha) = o(\alpha^2) \quad \text{as} \quad |\alpha| \to \infty \qquad (3.29)$$

Thus by an extension of Liouville's theorem,[12] (Sec. 2.52).

$$J(\alpha) = C_1\alpha + C_2 \quad (C_1,\ C_2 \text{ constants}) \qquad (3.30)$$

Equating each side of Eq. (3.28) to $J(\alpha)$, solving for Ψ_+' and Y_-', and adding, gives

$$(M\gamma^4 - L)(\Psi_+' + \Psi_-') = (M\gamma^4 - L)\Psi' = \left\{J(\alpha) - \frac{iAK_+(-b_o')}{\alpha + b_o'}\right\}\left(\frac{1}{K_+(\alpha)} - \frac{1}{K_-(\alpha)}\right)$$

*Note that this step is crucial to the ensuing argument. If we let $\beta = 1$ in condition (3.10) indicating an energy source at the origin, then all we can say is that Ψ_+, Ψ_+' are bounded as $|\alpha| \to \infty$ in D_+ and in the subsequent argument we find that $J(\alpha) = C_o\alpha^2 + C_1\alpha + C_2$. Then the solution is no longer unique; the strength of the energy source at the leading edge of the ice-field must be given in order to determine the additional constant.

From Eq. (3.13), this may be written in terms of Ψ . Thus,

$$\Psi(\alpha,H) = \left\{J(\alpha) - \frac{iAK_+(-b_o')}{\alpha+b_o'}\right\} \frac{1}{K_+(\alpha)\left\{K-\gamma \tanh \gamma H\right\}} \tag{3.31}$$

or

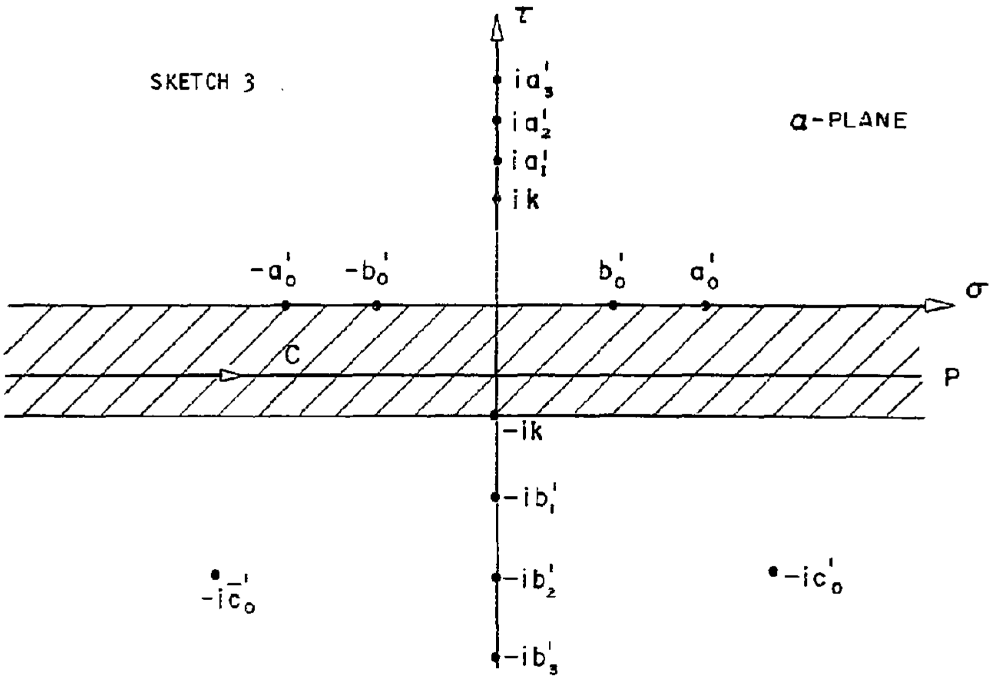
$$\Psi(\alpha,H) = \left\{J(\alpha) - \frac{iAK_+(-b_o')}{\alpha+b_o'}\right\} \frac{1}{K_-(\alpha)\left\{K-\gamma(1-L+M\gamma^4)\tanh \gamma H\right\}} \tag{3.32}$$

Equation (3.12) and the Fourier inversion formula give

$$\psi(y,z) = \frac{1}{2\pi}\int_C \Psi(\alpha,H)\,\frac{\cosh \gamma z}{\cosh \gamma H}\, e^{-i\alpha y} d\alpha \tag{3.33}$$

where C is some path in D as shown in Sketch 3.

SKETCH 3



The constants C_1 and C_2 will be determined from the conditions of zero shearing force and bending moment at the edge of the ice, $z = H$, $y = 0+$. Assume for the time being that this has been done. Then it is necessary to check that Eq. (3.33) with $\Psi(\alpha,H)$, given by either Eq. (3.31) or Eq. (3.32) does in fact satisfy the conditions of the problem. Now $\Psi(\alpha,h)$ is $O(\alpha^{-2})$ as $|\alpha| \to \infty$ in D so that the integral in Eq. (3.33) is uniformly convergent for $0 \le z \le H$, all y . Also, for $0 \le z < H$, all y , the integrals obtained by differentiating $\psi(y,z)$ with respect to y or z , any number of times are also uniformly convergent. Since the operator $\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - k^2$ applied to $\psi(y,z)$ makes the integrand vanish identically, it is clear that condition (3.3) is satisfied for $0 \le z < H$, all y . Similarly, condition (3.4) is seen to be satisfied on $z = 0$. To verify condition (3.5) for $y < 0$, Eq. (3.31) is used in Eq. (3.33), and the path of integration is deformed upwards in such a way that the ends of the path tend to infinity along lines in the upper half-plane. It is now permissible to apply the operator $K - \frac{\partial}{\partial z}$ to $\psi(y,z)$ and put $z = H$. Thus

$$\left(K\psi - \frac{\partial\psi}{\partial z}\right)_{z=H} = \frac{1}{2\pi}\int_{C'} \left\{J(\alpha) - \frac{iAK_+(-b_o')}{\alpha+b_o'}\right\} \frac{e^{-i\alpha y}}{K_+(\alpha)}\, d\alpha$$

where C' is the deformed path. It is clear that the only contribution to the integral arises from the only singularity of the integrand in D_+ , which is a pole at $\alpha = -b_o'$. Thus, for $y < 0$,

$$\left(K\psi - \frac{\partial\psi}{\partial z}\right)_{z=H} = Ae^{ib_o' y}$$

which verifies condition (3.5).

To verify condition (3.6) for $y > 0$, Eq. (3.32) is used in Eq. (3.33), and the ends of the path of integration are deformed downwards so as to tend to infinity along lines outside D in the lower half-plane. Now for $y > 0$, it is possible to differentiate $\psi(y,z)$ any number of times and put $z = H$. Thus

$$\left\{K - (1-L)\frac{\partial}{\partial z} - M\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 \frac{\partial}{\partial z}\right\} \psi(y,H)$$

$$= \frac{1}{2\pi}\int_{C''}\left\{J(\alpha) - \frac{iAK_+(-b_o')}{\alpha+b_o'}\right\}\frac{e^{-i\alpha y}}{K_-(\alpha)}\,d\alpha = 0$$

The boundedness of $\psi(y,z)$ everywhere follows from the fact that $\psi(y,z)$ is given by a uniformly convergent integral for $0 \leq z \leq H$, all y . A differentiation with respect to either y or z produces an additional α into the numerator of the integrand so that the discontinuous part of the integral behaves in a similar manner to

$$I = \int_o^\infty \frac{e^{-i\alpha y}e^{-\alpha(H-z)}}{\alpha + 1}\,d\alpha \sim \log r + \text{bounded terms near } z = H \text{ , } y = 0$$

Thus, condition (3.10) is verified.

Condition (3.7) is verified by using Eq. (3.32) in Eq. (3.33) and deforming the path of integration into the lower half-plane, taking into account the poles of the integrand. Since the singularity of the integrand nearest the real α-axis is at a distance greater than k from it, it is clear that ψ and also its first and second partial derivatives are $0(e^{-ky})$, for $y > 0$ and each z . A similar argument may be used to verify condition (3.8). In this case, contributions are obtained from the poles of the integrand at $\alpha = -b_o'$, $\alpha = \pm a_o'$, together with contributions of $0(e^{ky})$.

The conditions (3.17) and (3.18) require special attention. Now

$$\left(\frac{\partial\psi}{\partial z}\right)_{z=H} = \frac{1}{2\pi}\int_C F_1(\alpha)\,\frac{e^{-i\alpha y}\gamma\tanh\gamma H}{K_-(\alpha)\left[K-\gamma(1-L+M\gamma^4)\tanh\gamma H\right]}\,d\alpha$$

where

$$F_1(\alpha) = J(\alpha) - \frac{iAK_+(-b_o')}{\alpha+b_o'}$$

and where the path C has been deformed so as to pass beneath the point $\alpha = -ik$. Write

$$\frac{\gamma \tanh \gamma H}{K-\gamma(1-L+M\gamma^4)\tanh \gamma H} = \frac{1}{M\gamma^4}\left\{\frac{K-\gamma(1-L)\tanh \gamma H}{K-\gamma(1-L+M\gamma^4)\tanh \gamma H} - 1\right\}$$

Then $\dfrac{F_1(\alpha)}{M\gamma^4 K_-(\alpha)}$ is regular and tends to zero in $\mathrm{Im}\,(\alpha) < -k$, so that

$$\frac{1}{2\pi}\int_C \frac{F_1(\alpha)\, e^{-i\alpha y}}{M\gamma^4\, K_-(\alpha)}\, d\alpha = 0$$

Thus

$$\left(\frac{\partial\psi}{\partial z}\right)_H = \frac{1}{2\pi}\int_C \frac{F_1(\alpha)e^{-i\alpha y}\left\{K-\gamma(1-L)\tanh \gamma H\right\}}{M\gamma^4 K_-(\alpha)\left\{K-\gamma(1-L+M\gamma^4)\tanh \gamma H\right\}}\, d\alpha \tag{3.34}$$

where the integrand is $0(\frac{1}{\alpha^6})$ as $|\alpha| \to \infty$ in D.

It is clear that all y derivatives up to and including $\frac{\partial^3}{\partial y^3}(\frac{\partial\psi}{\partial z})_H$ are bounded for $y \geq 0$, and deforming the path of integration around the poles of the integrand indicates that all derivatives are $0(e^{-ky})$ for $y > 0$. It may be shown in a similar manner to the verification of condition (3.1) that

$$\frac{\partial^4}{\partial y^4}\left(\frac{\partial\psi}{\partial z}\right)_H = 0(\log r) \qquad r^2 = y^2 + (z-H)^2 \quad \text{near} \quad r = 0$$

Thus conditions (3.17) and (3.18) are verified, and $\psi(y,z)$ does indeed satisfy the conditions of the problem.

(b) The Far Field

In order to demonstrate the form of the solution for large y, it is convenient to use series representation of the solution.

Thus, for $y > 0$, the integral expression (3.33) may be written in terms of a sum of the residues at the poles of the integrands (see Appendix B). Thus,

$$\psi(y,z) = \frac{2F_1(-ic_o')e^{-c_o'y}c_o\cos c_oz\cos c_oH}{c_o'\left[2c_oH(1-L+Mc_o^4)+(1-L+5Mc_o^4)\sin 2c_oH\right]K_-(-ic_o')}$$

$$+\frac{2F_1(-i\bar{c}_o')e^{-\bar{c}_o'y}\bar{c}_o\cos\bar{c}_oz\cos\bar{c}_oH}{\bar{c}_o'\left[2\bar{c}_oH(1-L+M\bar{c}_o^4)+(1-L+5M\bar{c}_o^4)\sin 2\bar{c}_oH\right]K_-(-i\bar{c}_o')}$$

$$-\sum_{n=1}^{\infty}\frac{2F_1(-ib_n')e^{-b_n'y}b_n\cos b_nz\cos b_nH}{b_n'\left[2b_nH(1-L+Mb_n^4)+(1-L+5Mb_n^4)\sin 2b_nH\right]K_-(-ib_n')}$$

(3.35)

For $y < 0$, similarly,

$$\psi(y,z) = -e^{ib_o'y}\cosh b_oz$$

$$-\frac{2ia_o\cosh a_oz\cosh a_oH}{a_o'(2a_oH+\sinh 2a_oH)}\left\{\frac{F_1(a_o')e^{-ia_o'y}}{K_+(a_o')}-\frac{F_1(-a_o')e^{ia_o'y}}{K_+(-a_o')}\right\}$$

$$-\sum_{n=1}^{\infty}\frac{2F_1(ib_n')e^{b_n'y}b_n\cos b_nz\cos b_nH}{b_n'\left[2b_nH+\sin 2b_nH\right]}$$

(3.36)

Comparison of the coefficients in Eqs. (3.35) and (3.36) with the coefficients R and I occurring in Eqs. (3.1) and (3.2) indicates that

$$\frac{R}{I} = - \frac{F_1(+ a_o') \, K_+(- a_o')}{F_1(- a_o') \, K_+(+ a_o')} \tag{3.37}$$

and

$$(I)^{-1} = \frac{i a_o' \; (2a_o H + \sinh 2a_o H)}{2a_o \cosh a_o H} \quad \frac{K_+(- a_o')}{F_1(- a_o')} \tag{3.38}$$

The reflection coefficient $\mathcal{R}$ is defined as the ratio of the amplitude of the reflected wave to the incident wave at infinity. The transmission coefficient $\mathcal{T}$ is defined similarly.

Now

$$\frac{\partial \xi}{\partial t} = - \left(\frac{\partial \Phi}{\partial z}\right)_{z=H}$$

so that the elevation ξ is given by

$$\xi(x,y,t) = \mathrm{Re}\left\{ \frac{1}{i\omega}\left(\frac{\partial \phi}{\partial z}\right)_{z=H} \; e^{ikx - i\omega t} \right\}$$

Thus the elevation of the incident wave is

$$\xi(x,y,t) = \mathrm{Re}\left\{ \frac{a_o \sinh a_o H}{i\omega} I \; e^{ikx + ia_o' y - i\omega t} \right\} \tag{3.39}$$

with amplitude

$$I \; \frac{a_o \sinh a_o H}{\omega} \tag{3.40}$$

Similarly the amplitude of the reflected wave is

$$|R| \frac{a_o \sinh a_o H}{\omega} \tag{3.41}$$

The elevation of the transmitted wave is

$$\xi(x,y,t) = \text{Re}\left\{ \frac{b_o \sinh b_o H}{i\omega} e^{ikx + ib_o'y - i\omega t} \right\} \tag{3.42}$$

with amplitude $\dfrac{b_o \sinh b_o H}{\omega}$ so that the reflection coefficient is

$$\mathcal{R} = \frac{|R|}{|I|} = \left| \frac{F_1(a_o')}{F_1(-a_o')} \cdot \frac{K_+(-a_o')}{K_+(a_o')} \right| = \frac{|F_1(a_o')|}{|F_1(-a_o')|} \tag{3.43}$$

since

$$\left| \frac{K_+(-a_o')}{K_+(a_o')} \right| = 1 \qquad \text{(see Appendix B)}$$

and the transmission coefficient is

$$\mathcal{T} = \frac{b_o \sinh b_o H}{a_o \sinh a_o H} \cdot \frac{1}{|I|} = \frac{a_o'(2a_o H + \sinh 2a_o H)\sinh b_o H}{a_o^2 \sinh 2a_o H} \frac{|K_+(-a_o')|}{|F_1(-a_o')|} \tag{3.44}$$

(c) Determination of $J(\alpha)$

There are still two conditions which need to be satisfied at the leading edge of the ice before the solution is uniquely determined. These are conditions which express the fact that no energy is put in or taken out of the system at the leading edge of the ice-field. A derivation of these conditions is given by Hildebrand.[13] They are

$$D\frac{\partial^2\xi}{\partial y^2} + D\nu\,\frac{\partial^2\xi}{\partial x^2} = 0 \qquad (3.45)$$

$$D\,\frac{\partial}{\partial y}\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial x^2}\right) + D(1-\nu)\,\frac{\partial^3\xi}{\partial y\partial x^2} = 0 \qquad (3.46)$$

where

$- D\,\frac{\partial}{\partial y}\left(\frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial x^2}\right)$ is the transverse shearing force along $y = 0$

$(1-\nu)\,D\,\frac{\partial^2\xi}{\partial y\partial x}$ is the twisting moment along $y = 0$

$- D\left(\frac{\partial^2\xi}{\partial y^2} + \nu\,\frac{\partial^2\xi}{\partial x^2}\right)$ is the bending moment along $y = 0$

We have

$$\xi = \mathrm{Re}\left\{\frac{1}{i\omega}\,\frac{\partial\phi}{\partial z}\,e^{ikx-i\omega t}\right\}$$

and the conditions (3.45) and (3.46) in terms of ξ , become

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial\phi}{\partial z}\right)_{z=H} - \nu k^2\left(\frac{\partial\phi}{\partial z}\right)_{z=H} = 0 \quad , \; y = 0+ \qquad (3.47)$$

$$\frac{\partial^3}{\partial y^3}\left(\frac{\partial\phi}{\partial z}\right)_{z=H} - (2-\nu)k^3\,\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)_{z=H} = 0 \;, \quad y = 0+ \qquad (3.48)$$

These conditions, in turn, may be written in terms of $\psi(y,z)$. Thus

$$\begin{aligned}\frac{\partial^3}{\partial y^3}\left(\frac{\partial\psi}{\partial z}\right)_{z=H} - (2-\nu)k^2\,\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial z}\right)_{z=H} &= ib_ob_o'\sinh\,b_oH\,\left[b_o'^2 + (2-\nu)k^2\right]\\ &= ib_ob_o'\sinh\,b_oH\,\left[b_o^2 + (1-\nu)k^2\right]\end{aligned} \qquad (3.49)$$

and

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial\psi}{\partial z}\right)_{z=H} - \nu k^2\left(\frac{\partial\psi}{\partial z}\right)_{z=H} = b_o \sinh b_o H \left[b_o'^2 + \nu k^2\right]$$

$$= b_o \sinh b_o H \left[b_o^2 - (1-\nu)k^2\right] \quad y = 0+$$

(3.50)

In the case of normal incidence, k = 0 , and the conditions on ϕ are just

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial\phi}{\partial z}\right)_{z=H} = 0 \quad , \quad y = 0+ \tag{3.51}$$

$$\frac{\partial^3}{\partial y^3}\left(\frac{\partial\phi}{\partial z}\right)_{z=H} = 0 \quad , \quad y = 0+ \tag{3.52}$$

which compare with Stoker[6] .

Now if the expression (3.35) for $\psi(y,z)$ is differentiated with respect to z we obtain, for z = H ,

$$\left(\frac{\partial\psi}{\partial z}\right)_{z=H} = F_1(-ic_o')e^{-c_o'y}g(c_o)+F_1(-i\bar{c}_o')e^{-\bar{c}_o'y}g(\bar{c}_o)$$

$$+\sum_{n=1}^{\infty} F_1(-ib_n)e^{-b_n'y}\, g(b_n) \tag{3.53}$$

where

$$g(\beta) = \frac{2(K\cos\beta H + \beta(1-L)\sin\beta H)\cos\beta H}{M\beta^3 \cdot \beta' K_-(-i\beta')\,[2\beta H(1-L+M\beta^4) + (1-L+5M\beta^4)\sin 2\beta H]}$$

and

$$\beta' = +(\beta^2 - k^2)^{\frac{1}{2}}$$

Remember that

$$F_1(\alpha) = C_1\alpha + C_2 - \frac{iAK_+(-b_o')}{(\alpha+b_o')}$$

The conditions (3.49) and (3.50) when applied to (3.53) give

$$(c_o'^2-\nu k^2)F_1(-ic_o')g(c_o) + (\bar{c}_o'^2-\nu k^2)F_1(-i\bar{c}_o')g(\bar{c}_o)$$

$$+\sum_{n=1}^{\infty}(b_n'^2-\nu k^2)F_1(-ib_n)g(b_n) = b_o \sinh b_oH\left[b_o^2-(1-\nu)k^2\right] \qquad (3.54)$$

$$c_o'\left[c_o'^2-(2-\nu)k^2\right]F_1(-ic_o')g(c_o) + \bar{c}_o'\left[\bar{c}_o'^2-(2-\nu)k^2\right]F_1(-i\bar{c}_o')g(\bar{c}_o)$$

$$+\sum_{n=1}^{\infty} b_n'\left[b_n'^2-(2-\nu)k^2\right]F_1(-ib_n')g(b_n) = -ib_ob_o'\sin^2h\ b_oH\left[b_o^2-(1-\nu)k^2\right]$$

(3.55)

Now

$$F_1(\alpha) = C_2 + C_1\alpha - \frac{iAK_+(-b_o')}{\alpha+b_o'}$$

where

$$A = (Mb_o^4 - L)\ b_o \sinh\ b_oH$$

Clearly, there exist two equations for the determination of C_1 and C_2. After some algebra they may be written in the form

$$\begin{aligned} A_{11}\ C_1 + A_{12}\ C_2 &= B_1 \\ A_{21}\ C_1 + A_{22}\ C_2 &= B_2 \end{aligned} \qquad (3.56)$$

where

$$A_{11} = - i c_o'^2 \, [c_o'^2 - (2-\nu)\, k^2] \, g(c_o) - i \bar{c}_o'^2 \, [\bar{c}_o'^2 - (2-\nu)\, k^2] \, g(\bar{c}_o)$$

$$- i \sum_{n=1}^{\infty} b_n'^2 \, [b_n'^2 - (2-\nu)\, k^2] \, g(b_n) \tag{3.57}$$

$$A_{12} = c_o' \, [c_o'^2 - (2-\nu)\, k^2] \, g(c_o) + \bar{c}_o' \, [\bar{c}_o'^2 - (2-\nu)\, k^2] \, g(\bar{c}_o)$$

$$+ \sum_{n=1}^{\infty} b_n' \, [b_n'^2 - (2-\nu)\, k^2] \, g(b_n) \tag{3.58}$$

$$B_1 = - i b_o \, b_o' \sinh \, b_o H \, [b_o^2 - (1-\nu)\, k^2] - A K_+(- b_o') \left\{ \frac{c_o'}{c_o' + i b_o'} \, [c_o'^2 - (2-\nu) k^2] \, g(c_o \right.$$

$$\left. + \frac{\bar{c}_o'}{\bar{c}_o' + i b_o'} \, [\bar{c}_o'^2 - (2-\nu)\, k^2] \, g(\bar{c}_o) + \sum_{n=1}^{\infty} \frac{b_n'}{b_n' + i b_o'} \, [b_n'^2 - (2-\nu) k^2] \, g(b_n) \right\} \tag{3.59}$$

$$A_{21} = - i c_o' (c_o'^2 - \nu k^2) \, g(c_o) - i \bar{c}_o' \, (\bar{c}_o'^2 - \nu k^2) \, g(\bar{c}_c)$$

$$- i \sum_{n=1}^{\infty} b_n' (b_n'^2 - \nu k^2) \, g(b_n) \tag{3.60}$$

$$A_{22} = (c_o'^2 - \nu k^2) \, g(c_o) + (\bar{c}_o'^2 - \nu k^2) \, g(\bar{c}_o)$$

$$+ \sum_{n=1}^{\infty} (b_n'^2 - \nu k^2) \, g(b_n) \tag{3.61}$$

$$B_2 = b_o \sinh b_oH \, [b_o^2 - (1-\nu)k^2]$$

$$- AK_+(-b_o') \left\{ \frac{(c_o'^2 - \nu k^2)}{c_o' + ib_o'} g(c_o) + \frac{(\bar{c}_o'^2 - \nu k^2)}{\bar{c}_o' + ib_o'} g(\bar{c}_o) \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{(b_n'^2 - \nu k^2)}{(b_n' + ib_o')} g(b_n) \right\} \tag{3.62}$$

Once C_1 and C_2 have been determined, the solution of the problem is complete and, in particular, the expressions (3.43) and (3.44) for the reflection and transmission coefficients may be computed.

Unfortunately, the determination of the constants C_1 and C_2 above presents enormous computational difficulties, involving as it does the complex roots b_n of the transcendental equation

$$K \cos b_nH + b_n(1 - L + Mb_n^4) \sin b_nH = 0$$

Thus, whereas the solution to the full linear problem has been obtained, it will be necessary to resort to the shallow-water approximation to derive the solution in a form more suited to computation. It is possible, however, to derive a relation between $\mathcal{R}$ and $\mathcal{T}$ which will be done in the following section.

4. RELATION BETWEEN $\mathcal{R}$ AND $\mathcal{T}$

Although the explicit determination of $\mathcal{R}$ and $\mathcal{T}$ presents enormous computational difficulties, it is possible to derive a simple analytical relation between $\mathcal{R}$ and $\mathcal{T}$, by an application of Green's theorem. We have seen that the two-dimensional function $\phi(y,z)$ satisfies the following conditions (Eqs. 1.10-1.13).

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - k^2\phi = 0 \ , \ 0 < z < H \ , \ -\infty < y < \infty \tag{4.1}$$

$$\frac{\partial \phi}{\partial z} = 0 \ , \ z = 0 \ , \ -\infty < y < \infty \tag{4.2}$$

$$K\phi = \frac{\partial \phi}{\partial z} , \ z = H \ , \ -\infty < y < 0 \tag{4.3}$$

$$K\phi = (1-L)\frac{\partial \phi}{\partial z} + M\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 \frac{\partial \phi}{\partial z} \ , \ z = H, \ 0 < y < \infty \tag{4.4}$$

Also ϕ is continuous and satisfies the conditions (3.47) and (3.48), namely

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial \phi}{\partial z}\right)_{z=H} - \nu k^2 \left(\frac{\partial \phi}{\partial z}\right)_{z=H} = 0 \qquad y = 0+ \tag{4.5}$$

$$\frac{\partial^3}{\partial y^3}\left(\frac{\partial \phi}{\partial z}\right)_{z=H} - (2-\nu)k^2 \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right)_{z=H} = 0 \quad y = 0+ \tag{4.6}$$

Green's theorem applied to the functions $\phi(y,z)$ and its conjugate $\bar{\phi}(y,z)$ may be written

$$\iint_S (\phi \nabla^2 \bar{\phi} - \bar{\phi} \nabla^2 \phi)\, dy dz = \int_C \left(\phi \frac{\partial \bar{\phi}}{\partial n} - \bar{\phi} \frac{\partial \phi}{\partial n}\right) ds \tag{4.7}$$

where S is the area of the rectangle formed by the lines $z = 0$, $z = H$, and $y = \pm y_o$, C is the boundary of this rectangle, and n denotes the normal directed to the exterior of the boundary. Since ϕ and hence $\bar{\phi}$ satisfy Eq. (4.1), the left-hand side of (4.7) vanishes and the equation reduces to

$$\operatorname{Im} \int_C \phi \frac{\partial \bar{\phi}}{\partial n} \, ds = 0 \qquad (4.8)$$

It is assumed that y_o is so large that for $y = \pm y_o$, $\phi(y,z)$ may be replaced by its asymptotic form as given by Eqs. (3.1) and (3.2). Thus

$$\phi(y_o,z) \simeq e^{ib_o' y_o} \cosh \; b_c z \quad , \quad 0 \leq z \leq H \qquad (4.9)$$

$$\phi(-y_o,z) \simeq I e^{-ia_o' y_o} \cosh a_o z + R e^{+ia_o' y_o} \cosh a_o z \quad , \quad 0 \leq z \leq H \qquad (4.10)$$

Now the contribution to Eq. (4.8) from z = 0 vanishes by virtue of Eq. (4.2). Also, Eq. (4.3) implies that the contribution from z = H , $-\infty < y < 0$ vanishes. Thus Eq. (4.8) becomes

$$\operatorname{Im} \int_0^{y_o} \phi \frac{\partial \bar{\phi}(y,H)}{\partial z} dy + \operatorname{Im} \int_0^H (\phi \frac{\partial \bar{\phi}}{\partial y})_{y=y_o} dz - \operatorname{Im} \int_0^H (\phi \frac{\partial \bar{\phi}}{\partial y})_{y=-y_o} dz = 0 \qquad (4.11)$$

The first term presents the most difficulty, so this will be left to the last. Substituting the asymptotic form for ϕ into the second integral gives

$$\operatorname{Im} \int_0^H -ib_o' \cosh^2 b_o H \, dz = \frac{-b_o'}{4b_o}(2b_o H + \sinh 2b_o H)$$

Similarly for the final term in (4.11), we find

$$-\operatorname{Im} \int_0^H -ia_o' \left(I e^{-ia_o' y_o} + R e^{+ia_o' y_o}\right)\left(\bar{I} e^{+ia_o' y_o} - \bar{R} e^{-ia_o' y_o}\right) \cosh^2 a_o H \, dz$$

$$= a \frac{a_o'}{4a_o}(2a_o H + \sinh \; 2a_o H) \; (|I|^2 - |R|^2)$$

The first term in Eq. (4.11) may be written

$$\mathrm{Im}\,\frac{M}{K}\int_0^{y_o}\left(\frac{\partial^2}{\partial y^2}-k^2\right)^2\frac{\partial\phi}{\partial z}\,\frac{\partial\bar{\phi}(y,H)}{\partial z}\,dy$$

where Eq. (4.4) has been used, and

$$\mathrm{Im}\int_0^{y_o}\frac{\partial\phi}{\partial z}\,\frac{\partial\bar{\phi}}{\partial z}\,dy = 0$$

It is possible to integrate this expression by parts twice to obtain the following result

$$\mathrm{Im}\,\frac{M}{K}\int_0^{y_o}\left(\frac{\partial^2}{\partial y^2}-k^2\right)^2\frac{\partial\phi}{\partial z}\cdot\frac{\partial\bar{\phi}}{\partial z}\,dy$$

$$= \mathrm{Im}\,\frac{M}{K}\quad\left(\frac{\partial\bar{\phi}}{\partial z}\right)_H\left[\frac{\partial^3}{\partial y^3}\left(\frac{\partial\phi}{\partial z}\right)_H - (2-\nu)k^2\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)_H\right]_o^{y_o}$$

$$- \mathrm{Im}\,\frac{M}{K}\left[\nu k^2\left(\frac{\partial\bar{\phi}}{\partial z}\right)_H\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)_H + \frac{\partial}{\partial y}\left(\frac{\partial\bar{\phi}}{\partial z}\right)_H\frac{\partial^2}{\partial y^2}\left(\frac{\partial\phi}{\partial z}\right)_H\right]_o^{y_o}$$

where all the integrated terms were purely real and hence vanished when the imaginary part was taken.

Now the first term vanishes at $y = 0$ from condition (4.6) and the second term in brackets is seen to be purely real from Eq. (4.5) and so it also vanishes. The value of this expression at the top limit y_o may be determined, after some algebra, to be

$$\frac{-Mb'_o b_o^3\sinh 2b_oH}{1-L+Mb_o^4}$$

Combining the contributions from each of the terms in Eq. (4.8), we find that

$$\frac{-\,M b_o{}' b_o{}^3 \sinh 2b_oH}{1-L+Mb_o{}^4} - \frac{b_o{}'}{4b_o}(2b_oH + \sinh 2b_oH)$$

$$+\frac{a_o{}'}{4b_o}(2a_oH + \sinh 2a_oH)\left(|I|^2 - |R|^2\right) = 0$$

This may be written

$$\frac{b_o'}{b_o}\left[2b_oH+\sinh 2b_oH\left(1+\frac{4Mb_o{}^4}{1-L+Mb_o{}^4}\right)\right]\frac{1}{|I|^2} = \frac{a_o'}{a_o}\left(2a_oH + \sinh 2a_oH\right)\left(1 - \frac{|R|^2}{|I|^2}\right)$$

(4.12)

Now from Eqs. (3.43) and (3.44),

$$\mathcal{R}^2 = \frac{|R|^2}{|I|^2}$$

and

$$\mathcal{T}^2 = \frac{b_o{}^2 \sinh^2 b_oH}{a_o{}^2 \sinh^2 a_oH}\,\frac{1}{|I|^2}$$

$$\equiv \frac{b_o \sinh 2b_oH \quad 1}{a_o \sinh 2a_oH\,(1-L+Mb_o{}^4)}\cdot\frac{1}{|I|^2}$$

so that in terms of $\mathcal{T}^2$ and $\mathcal{R}^2$, Eq. (4.12) may be written

$$D\mathcal{T}^2 + \mathcal{R}^2 = 1 \tag{4.13}$$

where

$$D = \frac{a_o{}^2}{b_o{}^2}\cdot\frac{b_o'}{a_o'}\,\frac{\left[(1-L+Mb_o{}^4)2b_oH/\sinh 2b_oH + (1-L+5Mb_o{}^4)\right]}{\left[2a_oH/\sinh 2a_oH + 1\right]} \tag{4.14}$$

The Case $M = 0$

In this case

$$D = \frac{a_o^2}{b_o^2}\frac{b_o'}{a_o'}(1-L)\frac{\left[\frac{2b_oH}{\sinh 2b_oH} + 1\right]}{\left[\frac{2a_oH}{\sinh 2a_oH} + 1\right]} \tag{4.15}$$

where

$$K = a_o \tanh a_oH \quad , \quad \frac{K}{1-L} = b_o \tanh b_oH$$

so that

$$D = \frac{a_o^2 b_o'}{a_o' b_o^2}\frac{\left[(1-L)^2 b_o^2 H - K^2 H + K(1-L)\right]}{\left[a_o^2 H - K^2 H + K\right]} \tag{4.16}$$

which agrees with the value obtained by Weitz and Keller.[3]

5. SHALLOW-WATER APPROXIMATION

(a) Formulation and Solution

It is clear that the full solution of the linearized problem presents overwhelming numerical difficulties and it is desirable to examine the solution on the basis of the shallow-water approximation. Instead of considering the limit of the "finite depth" solution, it is desirable, for the sake of clarification, (and certainly easier) to rederive the solution from the original approximate equations based on the shallow-water theory.

We return to the Eqs. (1.1), (1.3), and (1.8) satisfied by the velocity potential $\Phi(x,y,z,t)$.

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0 \quad , \quad 0 < z < H \tag{5.1}$$

$$\Phi_z = 0 \quad , \quad z = 0 \tag{5.2}$$

$$\Phi_{tt} + g\Phi_z = 0 \quad , \quad z = H \ , \ -\infty < y < 0 \ , \ -\infty < x < \infty \tag{5.3}$$

$$\Phi_{tt} + g\Phi_z = -\frac{D}{\rho} \nabla^4 \Phi_z - \frac{\rho_i}{\rho} h \ \Phi_{ttz} \ , \ z = H \ , \ 0 < y < \infty$$

$$-\infty < x < \infty \tag{5.4}$$

We have seen that additional conditions at the leading edge of the ice-field are given by the equations

$$\frac{\partial}{\partial y} \left(\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial x \xi}\right) + (1-\nu) \frac{\partial^3 \xi}{\partial y \partial x^2} = 0 \qquad z = H \ , \ y = 0+ \tag{5.5}$$

$$\frac{\partial^2 \xi}{\partial y^2} + \nu \ \frac{\partial^2 \xi}{\partial x^2} = 0 \qquad z = H \ , \ y = 0+ \tag{5.6}$$

where $\xi(x,y,t)$ is the elevation of the ice. These equations ensure that no energy is put in or taken out of the ice at the leading edge. As before, it is convenient to write

$$\Phi(x,y,z,t) = \mathrm{Re} \left\{\phi(y,z) e^{ikx - i\omega t}\right\} \tag{5.7}$$

so that $\phi(y,z)$ satisifies

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - k^2 \phi = 0 \quad , \quad 0 < z < H \ , \ -\infty < y < 0 \tag{5.8}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad , \quad z = 0 \quad , \quad -\infty < y < \infty \tag{5.9}$$

$$K\phi = \frac{\partial\phi}{\partial z} \quad , \qquad z = H \quad , \quad -\infty < y < 0 \tag{5.10}$$

$$K\phi = (1-L)\,\frac{\partial\phi}{\partial z} + M\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2 \frac{\partial\phi}{\partial z} \quad , \quad z = H \quad , \quad 0 < y < \infty \tag{5.11}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial\phi}{\partial z}\right) - \nu k^2 \left(\frac{\partial\phi}{\partial z}\right) = 0 \quad , \qquad z = H \quad , \; y = 0+ \tag{5.12}$$

$$\frac{\partial^3}{\partial y^3}\left(\frac{\partial\phi}{\partial z}\right) - (2-\nu)k^2 \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) = 0 \quad , \quad z = H \; , \quad y = 0+ \tag{5.13}$$

In employing the shallow-water approximation the potential ϕ is expanded in powers of z, and it is found that

$$\phi(y,z) = \psi(y) - \frac{1}{2}\, z^2(\psi_{yy} - k^2\psi) \tag{5.14}$$

satisfies Eqs. (5.8) and (5.9) up to order z^3. A precise derivation of the theory is given by John[14] who concludes that the theory is valid provided the depth of the water is small compared to a wavelength and to the minimum radius of curvature of any immersed body. If the expression (5.14) for $\phi(y,z)$ is substituted into (5.10) and (5.11), it is found that $\psi(y)$ satisfies

$$\left(\frac{\partial^2}{\partial y^2} - k^2\right)\psi + K_o^2\psi = 0 \quad , \qquad -\infty < y < 0 \tag{5.15}$$

$$M\left(\frac{\partial^2}{\partial y^2} - k^2\right)^3\psi + (1-L)\left(\frac{\partial^2}{\partial y^2} - k^2\right)\psi + K_o^2\psi = 0 \quad , \quad 0 < y < \infty \tag{5.16}$$

where

$$K_o^2 = K/H = \frac{\omega^2}{gH} \quad , \quad L = \frac{\rho_i hHK_o^2}{\rho} \, , \; M = \frac{Eh^3}{12(1-\nu^2)\rho g}$$

in terms of $\psi(y)$ the conditions (5.12) and (5.13) become

$$\left(\frac{\partial^2}{\partial y^2} - \nu k^2\right)\left(\frac{\partial^2}{\partial y^2} - k^2\right) \psi(y) = 0 \; , \; y = 0+ \tag{5.17}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial y^2} - (2-\nu)k^2\right)\left(\frac{\partial^2}{\partial y^2} - k^2\right) \psi(y) = 0 \; , \; y = 0+ \tag{5.18}$$

In addition, from the shallow-water approximation,

$$\psi(y) \text{ and } \frac{\partial\psi(y)}{\partial y} \text{ are continuous at } y = 0 \tag{5.19}$$

Now the Eq. (5.15) has the solution $\psi(y) = e^{iK_o'y} + Re^{-iK_o'y}$ where $K_o' = (K_o^2 - k^2)^{\frac{1}{2}}$ and the positive square root is taken, and $K_o > k$ for a progressive wave. The form of the solution is such as to represent an incident wave of prescribed amplitude, and a reflected wave of (complex) amplitude R. Eq. (5.16) has a solution of the form $e^{ik_n'y}$ where

$$k_n' = (k_n^2 - k^2)^{\frac{1}{2}} \tag{5.20}$$

and the k_n $(n = 0, 1, \ldots 5)$ are the roots of the equation

$$Mk_n^6 + (1-L)k_n^2 = K_o^2 \tag{5.21}$$

This equation is a cubic equation in k_n^2 having solutions

$$k_o^2 = A + B \tag{5.22}$$

(Cont'd)

$$k_1^2 = \epsilon A + \epsilon^2 B$$
$$k_2^2 = \epsilon^2 A + \epsilon B \tag{5.22}$$

where

$$A = \left(\frac{K_o^2}{2M}\right)^{1/3} \left[1 + \left(1 + \frac{4(1-L)^3}{27MK_o^4}\right)^{1/2}\right]^{1/3} \tag{5.23}$$

$$B = \left(\frac{K_o^2}{2M}\right)^{1/3} \left[1 - \left(1 + \frac{4(1-L)^3}{27MK_o^4}\right)^{1/2}\right]^{1/3} \tag{5.24}$$

and

$$\epsilon = e^{2\pi i/3} = (-1 + \sqrt{3}i)/2$$

In all cases of practical interest $L < 1$ and $\frac{4(1-L)^3}{27MK_o^4} < 1$ so that $A + B > 0$.

Thus k_o^2 is real and positive while $\overline{k_2^2} = k_1^2$.

The square root in the expression $k_n' = (k_n^2 - k^2)^{\frac{1}{2}}$ is chosen such that $k_n' = k_n$ when $k = 0$. Now for $y > 0$ a solution is required representing a progressive wave. Clearly such a solution is $e^{ik_o'y}$. Other solutions are permitted provided they are bounded as $y \to +\infty$. Energy considerations exclude the possibility of unbounded solutions. The only possibilities are those roots k_n which have positive imaginary parts, so that $e^{ik_n'y}$ decays exponentially for increasing y. There are two such roots. Thus let $k_1 = +(k_1^2)^{\frac{1}{2}}$ have a positive imaginary part. Then $k_2 = -(k_2^2)^{\frac{1}{2}}$ also has a positive imaginary part, since $\overline{k_2} = -k_1$. Hence the solution for $y > 0$ may be written

$$\psi(y) = A_o e^{ik_o'y} + A_1 e^{ik_1'y} + A_2 e^{ik_2'y} \tag{5.25}$$

The unknowns are the complex constants A_0, A_1, A_2 and R, and there are just four conditions to be satisfied by $\psi(y)$. Since ψ and $\partial\psi/\partial y$ are continuous at $y = 0$, then the following equations must hold.

$$A_0 + A_1 + A_2 = 1 + R \tag{5.26}$$

$$k_0'A_0 + k_1'A_1 + k_2'A_2 = K_0'(1 - R) \tag{5.27}$$

Also, conditions (5.17) and (5.18) give the equations

$$\sum_{i=0}^{2} [k_i^2 - (1-\nu)k^2]\ A_i = 0 \tag{5.28}$$

$$\sum_{i=0}^{2} k_i'\ k_i^2\ [k_i^2 + (1-\nu)k^2]\ A_i = 0 \tag{5.29}$$

which together with Eqs. (5.26) and (5.27) are sufficient to determine the unknowns. The problem is solved once the constants A_0, A_1, A_2 and R have been computed, and expressions may be derived for physical quantities of interest.

(b) Reflection and Transmission Coefficients

The transmission coefficient is the ratio of the amplitude of the transmitted wave to the amplitude of the incident wave at infinity.

Now for large positive y,

$$\psi(y) \sim A_0 e^{ik_0'y} \tag{5.30}$$

since the other two terms in Eq. (5.25) decay exponentially with increasing y. The elevation $\xi(x,y,t)$ satisfies the Eq. (1.6), namely

$$\frac{\partial \xi}{\partial t} = - \frac{\partial \Phi}{\partial z} \quad \text{on} \quad z = H$$

and in terms of $\psi(y)$ the elevation may be written

$$\xi(x,y,t) = \text{Re}\left\{ \frac{iH}{\omega} k_o^2 A_o e^{ikx + ik_o' y - i\omega t} \right\} \tag{5.31}$$

and the corresponding amplitude is

$$\frac{Hk_o^2}{\omega} |A_o| \tag{5.32}$$

The incident wave is given $\psi = e^{iK_o' y}$ so that the elevation of the incident wave is

$$\xi(x,y,t) = - \text{Re}\left\{ \frac{iH}{\omega} K_o^2 e^{ikx + iK_o' y - i\omega t} \right\} \tag{5.33}$$

with amplitude $\frac{H}{\omega} K_o^2$. Note that this term contains the dimensional unit amplitude of the incident wave and hence it has the correct dimensions. Thus the transmission coefficient $\mathcal{T}$ is given by

$$\mathcal{T} = \frac{k_o^2}{K_o^2} |A_o| \tag{5.34}$$

and in a similar manner, the reflection coefficient $\mathcal{R}$ is given by

$$\mathcal{R} = |R| \tag{5.35}$$

(c) Pressure on the Bottom, $z = 0$

From Eq. (1.5) the pressure on the bottom in excess of atmospheric pressure p_o is

$$p = \rho \left(\frac{\partial \Phi}{\partial t}\right)_{z=0} + \rho g H \tag{5.36}$$

In terms of the shallow water potential $\psi(y)$, we have

$$p(x,y,0,t) = \rho \; \mathrm{Re} \left\{ -i\omega \psi(y) e^{ikx - i\omega t} \right\} + \rho g H \tag{5.37}$$

If the local effects represented by the exponentially decaying terms involving A_1 and A_2 in Eq. (5.25) are ignored, then the amplitude of the pressure fluctuation on the bottom under the ice is

$$p(x,y,0,t) - \rho g H = \mathrm{Re} \left\{ P e^{ikx + ik_o' y - i\omega t} \right\} \tag{5.38}$$

$$\text{where} \quad |P| = \rho\omega \, |A_o|$$

It is convenient to non-dimensionalize this in terms of the amplitude of the incident wave. Thus we define

$$\mathcal{P} = \frac{|P|}{\rho g \left(\frac{HK_o^2}{\omega}\right)} = \frac{\omega^2}{gHK_o^2} |A_o| = |A_o|$$

since $K_o^2 = \omega^2/gH$.

Hence $\mathcal{P}$ is a measure of the absolute value of the amplitude of the pressure fluctuation on the bottom $z = 0$, under the ice, compared to the amplitude of the pressure fluctuation on the bottom under the free surface.

(d) Relation Between $\mathcal{R}$ and $\mathcal{T}$ for Shallow Water

The Eqs. (5.26), (5.27), (5.28), (5.29) which determine the coefficients R and A_o conceal the relation existing between $\mathcal{R}$ and $\mathcal{T}$. This is most easily derived by taking the limit for small H of the relation which has been derived on the basis of the full linear equations.

From Eq. (4.13) we have

$$D\mathcal{T}^2 + \mathcal{R}^2 = 1$$

where

$$D = \frac{a_o^2}{b_o^2}\frac{b_o'}{a_o'}\frac{\left[(1-L+Mb_o^4)2b_oH/\sinh 2b_oH + (1-L+5M\,b_o^4)\right]}{\left[2a_oH/\sinh 2a_oH + 1\right]}$$

as

$$H \to 0 \quad , \quad \frac{2b_oH}{\sinh 2b_oH} \to 1 + O(b_oH)^2$$

so

$$D \to \frac{a_o^2}{b_o^2}\frac{b_o'}{a_o'}(1-L + 3Mb_o^4)$$

where b_o is the real positive roots of the equation

$$K = b_o^2(1-L + Mb_o^4)H$$

since $\tanh b_oH \sim b_oH(1 + O(b_oh)^2$.

This equation may be written

$$Mb_o^6 + (1-L)b_o^2 = \frac{K}{H} \quad \text{where} \quad \frac{K}{H} = \frac{\omega^2}{gH} = K_o^2$$

Clearly $b_o \to k_o$ in the shallow-water notation and $b_o' \to k_o'$. On the same basis

$$K = a_o^2 H \quad \text{so that} \quad a_o \to K_o \quad \text{and} \quad a_o' \to K_o'$$

Thus

$$D \to \frac{K_o^2}{k_o^2} \frac{k_o'}{K_o'} (1 - L + 3Mk_o^4)$$

and for shallow water

$$\frac{K_o^2 k_o'}{k_o^2 K_o'} (1 - L + 3Mk_o^4) \mathcal{T}^2 + \mathcal{R}^2 = 1$$

If we introduce the incident angle θ and a transmitted angle θ_T by writing

$$k = K_o \sin\theta = k_o \sin\theta_T$$

then the above formula becomes

$$\frac{K_o \cos\theta_T}{k_o \cos\theta} (1 - L + 3Mk_o^4) \mathcal{T}^2 + \mathcal{R}^2 = 1 \tag{5.39}$$

This relation will provide a check on the computed values of $\mathcal{R}$ and $\mathcal{T}$.

(e) The Critical Angle in Shallow Water

As in the case of finite depth of water, the constant k in the x-variation e^{ikx} determines the angle at which the incident wave enters the ice-field. Thus if we write $k = K_o \sin\theta$, then the incident wave is of the form

$$e^{iK_o y \cos\theta + iK_o x \sin\theta}$$

so that the incident wave makes an angle θ with the normal to the leading edge of the ice-field, as shown in Fig. 2.

The reflected wave is of the form

$$e^{-iK_o y \cos\theta + iK_o x \sin\theta}$$

indicating that the angle of incidence is equal to the angle of reflection. The transmitted wave has the form

$$e^{i(k_o^2 - K_o^2 \sin^2\theta)^{\frac{1}{2}} y + iK_o x \sin\theta} \equiv e^{ik_o \cos\theta_T y + ik_o \sin\theta_T x}$$

where k_o^2 satisfies

$$Mk_o^6 + (1-L)k_o^2 = K_o^2 \text{ , and } \frac{\sin\theta}{\sin\theta_T} = \frac{k_o}{K_o} \qquad (5.40)$$

Clearly when $\theta = 0$, $\sin\theta = 0$, and since k_o is real, there will always exist an undamped transmitted wave. (Assuming $L < 1$.) But if for particular values of M and L , $K_o > k_o$ then there will exist a critical value of θ , at which $k_o^2 = K_o^2 \sin^2\theta$ and above which $(k_o^2 - K_o^2 \sin^2\theta)^{\frac{1}{2}}$ is pure imaginary, thus producing an exponentially damped transmitted wave. This value of θ is given by $\theta_{crit} = \sin^{-1}(k_o/K_o)$. An incident wave approaching the ice at an incident angle greater than θ_{crit} will be totally reflected by the ice as shown in Fig. 3.

On the other hand, if $k_o > K_o$ then $(k_o^2 - K_o^2 \sin^2\theta)$ is always positive, so that waves approaching at any angle will penetrate the ice. In this case the transmitted wave is bent towards the normal so that the ice may be regarded as being "denser" than the water. When $K_o = k_o$, which, from Eq. (5.40), occurs when $MK_o^4 = L$, the transmitted wave proceeds at the same angle as the incident wave. The wavelength at which this occurs is denoted by λ_{crit} and hence any wave satisfying $\lambda > \lambda_{crit}$ penetrates the ice regardless of the incidence angle.

6. DISCUSSION OF RESULTS

(a) Description of Procedure

In obtaining the numerical results, the following values, taken from Robin,[7] were used

Young's modulus for ice	$E = 5\times10^{10}$ dyn/cm^2
Poisson's ratio for ice	$\nu = 0.3$
Density of ice	$\rho_i = 0.92$ gm/cm^3
Density of sea-water	$\rho = 1.025$ gm/cm^3

Now

$$K_o = \frac{2\pi}{\lambda}\text{ , so that }\ L = \frac{\rho_i}{\rho}\, hH\, \left(\frac{2\pi}{\lambda}\right)^2 = \frac{35.4Hh}{\lambda^2}$$

and

$$M = \frac{Eh^3}{12(1-\nu^2)\rho g} = 4.54 \times 10^6 h^3$$

Robin[7] has observed ice thicknesses of about 1.5 metres so that values of $h = 0.75$, 1.5, 3.0 metres were considered together with water depths of $H = 10$, 20 metres. The incidence angle θ was allowed to vary in steps of 15° from 0 to 60°, and various wavelengths of the incident wave, up to 700 metres, were considered.

In each of these cases, the Eqs. (5.26), (5.27), (5.28), and (5.29) with coefficients determined from Eqs. (5.22), (5.23), and (5.24) were solved and values were obtained for $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{P}$.

In addition, the critical angle θ_{crit} above which an incident wave is completely reflected and the critical wavelength, λ_{crit} above which an incident wave at any angle penetrates the ice, were computed in each case.

As a check on the numerical work, the expression

$$\frac{K_o \cos\theta_T(1-L+3Mk_o^{\,4})\mathcal{T}^2}{k_o\cos\theta} + \mathcal{R}^2 - 1$$

was evaluated. This should, of course, from Eq. (5.39) be identically zero. The fact that this expression was, in all cases, negligibly small provided a check on the numerical accuracy as well as a verification of the shallow-water approximation since the relation between $\mathcal{T}$ and $\mathcal{R}$ was derived by taking the limit for small H of the corresponding result based on the exact linear theory. Presumably this same relation could be derived directly from the analytic solution of the equations determining A_o and R.

The results are shown in Figs. 5 through 15.

(b) The Critical Angle

In Fig. 5, the critical incident angle is shown as a function of incident wavelength. The critical angle determines whether or not a given incident wave at a given incident angle is able to penetrate the ice (see Figs. 2 and 3). Thus a wave approaching the ice-field at an angle θ greater than the critical angle (as determined from Fig. 5) is completely reflected by the ice field. If the incident angle is less than θ_{crit} then penetration of the ice field occurs and an undamped transmitted wave travels through the ice field. Incident wavelengths greater than λ_{crit} corresponding to $\theta_{crit} = \frac{\pi}{2}$ will always penetrate the ice, regardless of the incident angle (see Fig. 4).

(c) Reflection and Transmission Coefficients

Figures 6 to 10 show the variation of $\mathcal{R}$ and $\mathcal{T}$ with the incident wavelength λ for ice thicknesses of 0.75, 1.5, and 3.0 metres, incident angles ranging from 0^o to 60^o in steps of 15^o, and water depths of 10 and 20 metres.

In Fig. 6 the curves for $\mathcal{T}$ have been extrapolated so as to pass through the origin. This is not strictly accurate since at $\lambda = 0$, L is infinite, and it has been assumed that $L < 1$. There does in fact exist a cut-off wavelength (unrelated to the incident angle) which occurs whenever Eq. (5.21) fails to have a real root. This occurs when $\frac{4(L-1)^3}{27MK_o^4} > 1$ and then all possible solutions in the ice-field decay exponentially with distance. This restriction is of little physical significance since, for example, with $h = 1.5$ metres, $H = 10$ metres, no propagation is possible for $\lambda < 10^{-4}$ cms, approximately. In other words, this mathematical phenomenon only occurs at wavelengths too small to be physically significant.

Figures 6 to 10 illustrate the importance of ice thickness in determining reflection and transmission. Thus, for instance, from Fig. 6, doubling the ice thickness from 1.5 to 3 metres causes a drop of over 25% in the wave amplitude of the transmitted wave corresponding to an incident wavelength of 250 metres. Since the wave energy is proportional to the square of the wave amplitude, this means almost a 50% reduction in energy caused by doubling the ice thickness.

In contrast, the water depth has little effect on the reflection and transmission coefficients except for the smaller wavelengths.

As would be expected, the ability of the ice to reflect the incident wave is reduced as the incident wavelength increases, and for $\lambda > 500$ metres, the transmitted wave height is at least 95% of the incident wave height. In all cases, the amount of reflection was very small except for the lower wavelengths. This agrees with the conclusions of Stoker,[6] who considered the two-dimensional problem ($\theta = 0^{o}$).

Figures 6 to 10 furthermore indicate the effect of the angle of incidence upon the reflection and transmission coefficients. For a fixed wavelength there appears to be a gradual increase in transmission with increasing incident angle. For a given incident angle $\mathcal{T}$ decreases until the cut-off wavelength is reached, at which $\mathcal{T}$ drops to zero and complete reflection occurs. For larger incident angle a distortion of the transmission curves takes place so that, for example, when $\theta = 60^{o}$, the transmission curve for ice of 3 metres thickness has a minimum at a wavelength of about 320 metres.

(d) Pressure Amplitude on the Bottom Under the Ice

Figures 11 to 15 show the variation in non-dimensional pressure fluctuations on the bottom under the ice normalized with respect to the pressure on the bottom in the absence of ice, in terms of the incident wavelength λ , ice thickness h , water depth H , and incident angle θ .

In all cases the pressure amplitude $\mathcal{P}$ tends to unity as λ increases, which is to be expected. For $\theta = 0^{\circ}$, the pressure is generally lower than the corresponding pressure amplitude on the bottom in the absence of ice, the pressure drop being greater for greater ice thickness. For $\theta = 15^{\circ}$, considerable changes in the pressure curves take place and for the lower incident wavelengths $\mathcal{P}$ is actually greater than unity for an ice thickness of 3 metres. This trend increases for larger values of θ so that for $\theta = 60^{\circ}$ the pressure jumps to over 1.5 the free surface bottom pressure in the most extreme cases. As in the case of the transmission coefficients, for each non-zero angle there is a cut-off wavelength corresponding to θ_{crit} at which the pressure drops abruptly to zero, indicating complete reflection of the incident wave.

(e) Comparison of Results With Observation

Robin[7] has concluded that no effective transmission of wave energy occurs for wavelengths less than 200 metres in fields of large floes, whereas major energy changes occurred in those waves having a period of 16 seconds, or wavelengths of 400 metres.

It is difficult to compare observations in deep water with a theory based on the shallow water approximation. Thus Figs. 6 to 10 indicate that energy transmission through the ice occurs at much lower wavelengths than observed by Robin.[7] For instance, when λ = 100 metres, h = 3.0m, $\mathcal{R}$ = 0.3 indicating that over 90% of the incident energy has been transmitted into the ice. This discrepancy is almost certainly due to using the shallow-water approximation. If the numerical difficulties involved in the solution based on the full linearized theory could be surmounted, it is felt that a much closer comparison of theory and observation would result.

The experimental counterpart of this wave-ice investigation is reported in Ref. 15.

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B. MODEL III. AN ICE-FIELD HAVING SURFACE TENSION

1. FORMULATION AND SOLUTION

In this model it is assumed that each element of the ice sheet is subjected to a force arising from a surface tension force in the sheet. Then it may be shown that

$$p - p_o = - T_s \nabla^2 \xi + \rho_i h \xi_{tt} \qquad (6.1)$$

where T_s = surface tension force.

The theory proceeds exactly as for Model II except that the free surface condition satisfied by ψ becomes

$$K\psi = \frac{(1-L)\partial\psi}{\partial z} - S\left(\frac{\partial^2}{\partial y^2} - k^2\right) \frac{\partial\psi}{\partial z} \quad , \quad z = H \, , \, 0 < y < \infty \qquad (6.2)$$

where $S = T_s/\rho g$.

The characteristic equation is

$$K \cos b_n H + b_n \left\{1-L-Sb_n^2\right\} \sin b_n H = 0 \qquad (6.3)$$

which has an infinite sequence of real roots $\pm b_n (n = 1,2, \ldots)$ and two pure imaginary roots $\pm ib_o$, for $K,S > 0$, $L < 1$. Thus propagation always occurs in the ice, if $|b_o| > k$. Also, complete reflection of a plane incident wave will always occur for some angle of incidence whenever $L < Sa_o^2$.

The transform of condition (6.2) is obtained by integrating by parts; thus

$$K\Psi_+ = \Psi_+'(1-L + S\gamma^2) + S(a_2 - i\alpha a_1) \quad (\alpha \in D_+) \qquad (6.4)$$

with the same notation as before. Similarly, for $y < 0$

$$K\Psi_- = \Psi_-' \quad (\alpha \in D_-) \tag{6.5}$$

Following the same procedure as in Model II, we arrive at the equation

$$K_+(\alpha)\left\{(S\gamma^2-L)\Psi_+' - S(a_2-i\alpha_1)\right\} - iA\left\{\frac{K_+(\alpha)-K_+(-b_o')}{\alpha+b_o'}\right\}$$

$$= -K_-(\alpha)\left\{(S\gamma^2-L)\Psi_-'-S(a_2-i\alpha a_1)\right\} - iA\left\{\frac{K_-(\alpha)-K_+(-b_o')}{\alpha+b_o'}\right\} = J(\alpha)$$

$$(\alpha \in D) \tag{6.6}$$

where

$$\frac{K\cosh\gamma H-\gamma(1-L+S\gamma^2)\sinh\gamma H}{K\cosh\gamma H-\gamma\sinh\gamma H} = \frac{K_+(\alpha)}{K_-(\alpha)}$$

and $K_\pm(\alpha)$ is regular and non-zero in $D_\pm$, respectively.

The left-hand side of (6.6) is regular everywhere in D_+ while the right-hand side is regular everywhere in D_-. The two sides are equal in the strip D. This defines a function $J(\alpha)$ regular in the whole α-plane. It may be shown that

$$K_+(\alpha) = 0(\alpha) \qquad |\alpha| \to \infty \text{ in } D_+$$

$$K_-(\alpha) = 0(\alpha^{-1}) \qquad |\alpha| \to \infty \text{ in } D_-$$

Furthermore $\Psi_\pm' = o(1)$, $|\alpha| \to \infty$ in $D_\pm$, as in Model II.

But the left-hand side of (6.6) is just

$$K_+(\alpha)\left\{K\Psi_+ - \Psi_+'\right\} - iA\left\{\frac{K_+(\alpha) - K_+(-b_o')}{a + b_o'}\right\} = J(\alpha) \tag{6.7}$$

and so $J(\alpha) = o(\alpha)$ as $|\alpha| \to \infty$ in D_+ .

Similarly from the right-hand side of (6.6)

$$J(\alpha) = o(\alpha) \text{ , as } |\alpha| \to \infty \text{ in } D_-$$

Thus from an extension of Liouville's theorem

$$J(\alpha) = C \text{ , a constant}$$

We find from Eq. (6.6), that

$$\Psi(\alpha,H) = \frac{C - iAK_+(-b_o')}{\alpha + b_o'} \cdot \frac{1}{(K - \gamma\tanh\gamma H)K_+(\alpha)}$$

and so

$$\psi(y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\alpha,H)\,\frac{\cosh\gamma z}{\cosh\gamma H}\, e^{-i\alpha y} d\alpha$$

where the path of integration is in D .

We see that the solution depends upon a constant C . An additional condition is required to determine C .

The required condition is obtained by applying Green's theorem to the function $\phi(y,z)$ and its complex conjugate $\bar{\phi}$ in the rectangle formed by the lines $z = 0$, $z = H$, $y = \pm y_o$. Note, by analogy with Model II, that $\phi(y,z)$ satisfies

$$(\partial^2/\partial y^2 + \partial^2/\partial z^2)\phi(y,z) = 0 \tag{6.8}$$

$$K\phi = \frac{\partial\phi}{\partial z} \text{ , } z=H, \ -\infty < y < 0 \tag{6.9}$$

$$K\phi = (1-L)\frac{\partial\phi}{\partial z} - S\left(\frac{\partial^2}{\partial y^2} - k^2\right)\frac{\partial\phi}{\partial z}, z=H, \ 0 < y < \infty \qquad (6.10)$$

where $K = \frac{\omega^2}{g}$, $L = \frac{\rho_i h\omega^2}{\rho g}$, $S = \frac{T_s}{\rho g}$

$$\phi(y,z) = T e^{ib_o'y} \cosh b_o z + O(e^{-ky}), \ 0 \le z \le H, \ y > 0 \qquad (6.11)$$

$$\phi(y,z) = I e^{ia_o'y} \cosh a_o z + R e^{-ia_o'y} \cosh a_o z + O(e^{ky}) \qquad (6.12)$$
$$0 \le z \le H, \ y < 0$$

where

$$a_o' = (a_o^2 - k^2)^{\frac{1}{2}} , \ b_o' = (b_o^2 - k^2)^{\frac{1}{2}}$$

and

$$K \cosh a_o H = a_o \sinh a_o H$$

$$K \cosh b_o H = b_o (1-L+Sb_o^2) \sinh b_o H$$

The method is identical to that used in Model II to determine the relation between $\mathcal{R}$ and $\mathcal{T}$. In addition, it provides us with the required conditions for the potential at infinity to be unique.

Green's theorem reduces to

$$\mathrm{Im} \int_C \phi \frac{\partial\bar{\phi}}{\partial n} ds \qquad (6.13)$$

where the line integral is taken over the boundary of the rectangle. The contributions from $z = 0$, and from $z = H$, $-\infty < y < 0$ vanish, and the contribution from $z = H$, $0 < y < \infty$ is

$$\mathrm{Im}\int_0^{y_0} \phi\,\frac{\partial\bar{\phi}}{\partial z}\,dy = \mathrm{Im}\,\frac{-S}{K}\int_0^{y_0}\left(\frac{\partial^2}{\partial y^2} - k^2\right)\left(\frac{\partial\phi}{\partial z}\right)_{z=H}\left(\frac{\partial\bar{\phi}}{\partial z}\right)_{z=H} dy$$

$$= \frac{-S}{K}\,\mathrm{Im}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)_{z=H}\left(\frac{\partial\bar{\phi}}{\partial z}\right)_{z=H}\right]_{y=0+}^{y=y_0}$$

$$= \left[\frac{-Sb_o^2\sinh^2 b_oHb'_o}{K} + \frac{S}{K}\quad \mathrm{Im}\left(\frac{\partial^2\phi}{\partial y\partial z}\,\frac{\partial\bar{\phi}}{\partial z}\right)_{\substack{z=H\\y=0+}}\right]|T|^2$$

The contributions to Eq. (6.13) from the boundaries $y = \pm y_o$ are

$$\frac{a'_o}{4a_o}\,(2a_oH+\sinh 2c_oH)\,(|I|^2-|R|^2)$$

and

$$\frac{-b'_o}{4b_o}\,(2b_oH+\sinh 2b_oH)\,|T|^2 \quad \text{, respectively}$$

Thus combining all the terms, and using the relation

$$\sinh^2 b_oH = \frac{K\sinh 2b_oH}{2b_o(1-L+Sb_o^2)}$$

we obtain

$$|T|^2\,\frac{b'_o}{b_o}\left[2b_oH+\sinh 2b_oH\left(1+\frac{2Sb_o^2}{1-L+Sb_o^2}\right)\right] + \frac{4S}{K}\,\mathrm{Im}\left(\frac{\partial^2\phi}{\partial y\partial z}\,\frac{\partial\bar{\phi}}{\partial z}\right)_{\substack{z=H\\y=0+}}$$

$$= \frac{a'_o}{a_o}\,(2a_oH+\sinh 2a_oH)\,(|I|^2-|R|^2) \qquad (6.14)$$

Now from Eq. (6.14), if $|I|^2 \equiv 0$, so that no energy is being supplied to the system, $R = T = 0$ if and only if

$$\mathrm{Im}\left(\frac{\partial^2 \phi}{\partial y \partial z}\frac{\partial \bar{\phi}}{\partial z}\right)_{\substack{z=H\\ y=0+}} = 0 \tag{6.15}$$

This condition must therefore be satisfied to ensure that no energy is being supplied to the system other than by the incident wave. Then the following relation may be derived between the reflection coefficient

$$\mathcal{R}^2 = \frac{|R|^2}{|I|^2} \tag{6.16}$$

and the transmission coefficient

$$\mathcal{T}^2 = \frac{b_o{}^2 \sinh^2 b_o H}{a_o{}^2 \sinh^2 a_o H}\;\frac{|T|^2}{|I|^2} \tag{6.17}$$

$$D\mathcal{T}^2 + \mathcal{R}^2 = 1 \tag{6.18}$$

where

$$D = \frac{a_o{}^2}{b_o{}^2}\,\frac{b_o'}{a_o'}\,\frac{\left[(1-L+Sb_o{}^2)2b_oH/\sinh 2b_oH + (1-L+3Sb_o{}^2)\right]}{\left[2a_oH/\sinh 2a_oH + 1\right]} \tag{6.19}$$

When H is small, this reduces to

$$D = \frac{K_o{}^2 k_o'}{k_o{}^2 K_o'}\,(1-L+2Sk_o{}^2) \tag{6.20}$$

in the notation of the shallow-water approximation used in Model II. The condition (6.15) may be satisfied in more than one way and the physical unreality of the model does not provide an obvious choice. However, the tension existing in the ice sheet, $z = H$, $y > 0$ suggests there must be a restraining force at the edge of the sheet $y = 0+$. For instance, if the ice is fixed at that point in some way such that the elevation is zero, then, $\left(\frac{\partial \phi}{\partial z}\right)_{\substack{z=H \\ y=0+}} = 0$ and condition (6.15) is satisfied. This condition is sufficient to determine the constant C so that the solution is uniquely determined.

This work provides an extension of the problem considered by Keller and Goldstein[5] who utilized the shallow-water approximation. The authors apply a different third condition at the edge of the ice-field in order to fully determine the solution. This condition which is essentially the continuity of vertical velocity at the leading edge, is not sufficient to ensure that no energy is fed into the system other than from the incident wave, and also, the authors do not discuss the instability of the stretched membrane at the edge $z = H$, $y = 0+$.

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APPENDIX A

It is required to find the position of the roots of the equation

$$K\cos b_n H + b_n \left\{1-L+Mb_n^4\right\} \sin b_n H = 0,\ L < 1,\ M,\ K > 0 \tag{A-1}$$

By sketching the curves $\tan b_n H$ and $-\dfrac{K}{b_n \left\{1-L+Mb_n^4\right\}}$, it may be shown that there is an infinite sequence of real zeros $\pm b_n (n = 1,2,\ldots)$ such that $|b_n| < |b_{n+1}|$, and furthermore

$$b_n \sim \frac{n\pi}{H} + 0\left(\frac{1}{n^5}\right) \text{ as } n \to \infty \tag{A-2}$$

The intersection of the curves $\tanh b_n H$ and $\dfrac{K}{b_n(1-L+Mb_n^4)}$ indicate there are also two pure imaginary roots $\pm i b_o$.

The number and position of the complex roots of (A-1) are determined by considering the equivalent equation

$$A + z(1+Bz^4)\tan z = 0 \tag{A-3}$$

for complex $z(= x + iy)$.

Now

$$\left|\frac{A}{z(1+Bz^4)}\right| \to 0 \text{ as } |z| \to \infty$$

whereas

$$|\tan z| = \left(\frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}\right)^{\frac{1}{2}} \to 0$$

provided $x \to n\pi$ and $y \to 0$, simultaneously.

Equation (A-3) may be split into real and imaginary parts and one form of the resulting equations is

$$\frac{5(d_1 x^2 - y^2)(d_2 x^2 - y^2)}{(e_1 x^2 - y^2)(e_2 x^2 - y^2)} \quad \frac{\sinh 2y}{2y} = - \frac{\sin 2x}{2x} \tag{A-4}$$

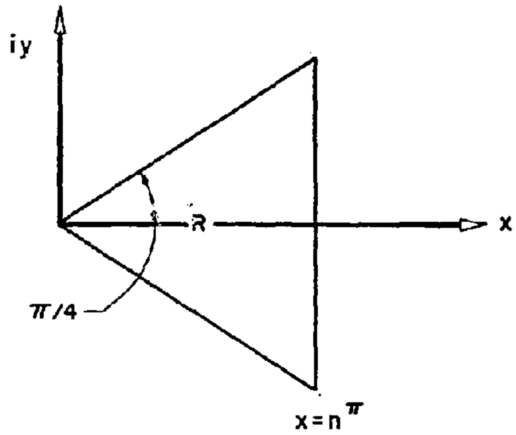
$$A\left(\frac{\sinh 2y}{2y} + \frac{\sin 2x}{2x}\right) = 2B(x^4 - y^4)\ (\cosh 2y - \cos 2x) \tag{A-5}$$

where

$$d_{1,2} = 1 \pm \frac{1}{5}\sqrt{20 - 5/Bx^4}$$

$$e_{1,2} = 5 \pm \sqrt{20 - 1/Bx^4} \quad .$$

Now as $x \to n\pi$, n large, $y \to 0$, the left-hand side of (A-4) tends to 1, while the right-hand side tends to zero. Hence, there exists no solution to (A-3) for $x > x_o$ for some $x_o = x_o(A,B)$. Also, it is clear from (A-5) that $x > y$ for a solution to exist. Now from (A-3), if z_o is a solution with $o < \arg z < \frac{\pi}{4}$, then $-z_o$, $\bar{z}_o$ and $-\bar{z}_o$ are also solutions. There are thus a finite number of solutions of (A-2) occurring in pairs within the region R bounded by the lines $y = \pm x$, $x = n\pi$, for n large enough. (See sketch on the following page).



The number of roots lying in R may be determined by the method of the argument. Thus, if

$$f(z) = A + z(1+Bz^4)\tan z \quad ,$$

$$\frac{1}{2\pi}\,\Delta \arg f(z) = (\text{no. of zeros}) - (\text{no. of poles})$$

where $\Delta \arg f(z)$ means the change in the argument of $f(z)$ as z traverses the boundary of R. It may be shown that $\frac{1}{2\pi}\Delta \arg f(z) = 2$ and since there are n real zeros and n poles in R, then there must also be two complex zeros. Thus, in the entire plane there are four complex zeros of $f(z)$, such that $x > y$.

Thus, Eq. (A-1) has complex roots $\pm c_o$, $\pm \bar{c}_o$ where $c_o = \lambda + i\mu$ and $\lambda > \mu > 0$.

In a similar manner, it may be shown that the equation

$$K \cos a_n H + a_n \sin a_n H = 0 \tag{A-6}$$

has an infinite sequence of real zeros $a_n (n = 1,2,\cdots)$ such that $|a_n| < |a_{n+1}|$, and furthermore

$$a_n \sim \frac{n\pi}{H} + O\left(\frac{1}{n}\right) \text{ as } n \rightarrow \infty \qquad \text{(A-7)}$$

There are also two pure imaginary roots $\pm ia_o$. It may be shown that there are no complex roots by considering the equation

$$A + z\tan z = 0 \quad \text{for complex} \quad z = x+iy \qquad \text{(A-8)}$$

When (A-8) is split into real and imaginary parts, it is found that one of the resulting equations may be written

$$\frac{\sinh 2y}{2y} + \frac{\sin 2x}{2x} = 0 \qquad \text{(A-9)}$$

Now $\left|\frac{\sinh 2y}{2y}\right| > 1$ whereas $\left|\frac{\sin 2x}{2x}\right| < 1$

so that (A-9) has no solution for $x, y \neq 0$.

APPENDIX B

It is required to determine functions $K_{\pm}(\alpha)$ such that

$$\frac{f_1(\alpha)}{f_o(\alpha)} = K_+(\alpha)/K_-(\alpha)$$

and $K_{\pm}(\alpha)$ is regular and non-zero in $D_{\pm}$.

Consider the function

$$f_1(\alpha) = K\cosh\gamma H - \gamma(1-L+M\gamma^4)\sinh\gamma H$$

This has zeros $\gamma = \pm b_o$, $\gamma = \pm ib_n (n = 1,2,\ldots)$ and $\gamma = \pm ic_o$, $\pm i\bar{c}_o$, Re $c_o > 0$. Thus, since $\gamma^2 = \alpha^2 + k^2$, $\alpha = \pm b_o'$, $\pm ib_n'$, $\pm ic_o'$, $\pm i\bar{c}_o'$, where Re $c_o' > k$ since $0 < \arg c_o < \frac{\pi}{4}$.

Since $f_1(\alpha)$ is an entire function (a function whose only singularities are poles), it may be written as an infinite product. (Noble[11], p.15) Thus

$$f_1(\alpha) = K\left(1 - \frac{\gamma^2}{b_o^2}\right)\left(1 + \frac{\gamma^2}{c_o^2}\right)\left(1 + \frac{\gamma^2}{\bar{c}_o^2}\right)\prod_{n=1}^{\infty}\left(1 + \frac{\gamma^2}{b_n^2}\right) \quad (n = 1,2,\ldots)$$

where $\gamma^2 = \alpha^2 + k^2$.

In a similar manner, the function $f_o(\alpha) = K\cosh\gamma H - k\sinh\gamma H$ may be written in the form

$$f_o(\alpha) = K\left(1 - \frac{\gamma^2}{a_o^2}\right)\prod_{n=1}^{\infty}\left(1 + \frac{\gamma^2}{a_n^2}\right)$$

Once the singularities of $f_o(\alpha)$, $f_1(\alpha)$ are revealed in this manner, it is possible to define $K_\pm(\alpha)$ in the following way.

Let

$$K_+(\alpha) = \left\{(1+\frac{k^2}{c_o^{\,2}})^{\frac{1}{2}} - \frac{i\alpha}{c_o}\right\}\left\{(1+\frac{k^2}{\bar{c}_o^{\,2}})^{\frac{1}{2}} - \frac{i\alpha}{\bar{c}_o}\right\} \prod_{n=1}^{\infty} \frac{\left\{(1+k^2/b_n^{\,2})^{\frac{1}{2}} - \frac{i\alpha}{b_n}\right\}}{\left\{(1+k^2/a_n^{\,2})^{\frac{1}{2}} - \frac{i\alpha}{a_n}\right\}}$$

$$K_-(\alpha) = \frac{\left\{1 - \frac{(\alpha^2+k^2)}{a_o^{\,2}}\right\}}{\left\{1-\frac{(\alpha^2+k^2)}{b_o^{\,2}}\right\}\left\{(1+\frac{k^2}{c_o^{\,2}})^{\frac{1}{2}} + \frac{i\alpha}{c_o}\right\}\left\{(1+\frac{k^2}{\bar{c}_o^{\,2}})^{\frac{1}{2}} + \frac{i\alpha}{\bar{c}_o}\right\}} \prod_{n=1}^{\infty} \frac{\left\{(1+\frac{k^2}{a_n^2})^{\frac{1}{2}} + \frac{i\alpha}{a_n}\right\}}{\left\{(1+\frac{k^2}{b_n^2})^{\frac{1}{2}} + \frac{i\alpha}{b_n}\right\}}$$

Then

$$\frac{K_+(\alpha)}{K_-(\alpha)} = \frac{K\cosh\gamma H - \gamma(1-L+M\gamma^4)\sinh\gamma H}{K\cosh\gamma H - \gamma\sinh\gamma H}$$

and $K_\pm(\alpha)$ is regular and non-zero in $D_\pm$, respectively.

It is noteworthy that

$$K_+(\alpha)K_-(-\alpha) = \frac{\left\{1-\frac{(\alpha^2+k^2)}{a_o^{\,2}}\right\}}{\left\{1-\frac{(\alpha^2+k^2)}{b_o^{\,2}}\right\}} \quad \text{and that} \quad \left|\frac{K_+(-a_o')}{K_+(a_o')}\right| = 1$$

The behavior of $K_\pm(\alpha)$ as $|\alpha| \to \infty$ in $D_\pm$ may be determined by considering the behavior of the infinite products as $|\alpha| \to \infty$. Now it may be shown that

$$a_n = \frac{n\pi}{H} + O(\frac{1}{n}) \quad \text{as} \quad n \to \infty$$

whereas

$$b_n = \frac{n\pi}{H} + 0(\frac{1}{n^5}) \quad \text{as} \quad n \to \infty$$

Thus

$$\prod_{n=1}^{\infty} \frac{\left\{(1+\frac{k^2}{b_n^2})^{\frac{1}{2}} - \frac{i\alpha}{b_n}\right\}}{\left\{(1+\frac{k^2}{a_n^2})^{\frac{1}{2}} - \frac{i\alpha}{a_n}\right\}} = \prod_{n=1}^{\infty} (\frac{a_n}{b_n}) \prod_{n=1}^{\infty} \left\{\frac{\alpha+i(b_n^2 + k^2)^{\frac{1}{2}}}{\alpha+i(a_n^2 + k^2)^{\frac{1}{2}}}\right\}$$

Now

$$\frac{a_n}{b_n} = 1 + 0(\frac{1}{n^2}) \quad \text{so that} \quad \prod_{n=1}^{\infty} (\frac{a_n}{b_n}) = \text{constant} < \infty$$

Also

$$\prod_{n=1}^{\infty} \left\{\frac{\alpha+i(b_n^2 + k^2)^{\frac{1}{2}}}{\alpha+i(a_n^2 + k^2)^{\frac{1}{2}}}\right\} = \prod_{n=1}^{\infty} \left\{1 + f_n(\alpha)\right\}$$

where

$$f_n(\alpha) = \frac{(b_n^2 + k^2)^{\frac{1}{2}} - (a_n^2 + k^2)^{\frac{1}{2}}}{(a_n^2 + k^2)^{\frac{1}{2}} - i\alpha}$$

Clearly $f_n(\alpha) \to 0$ as $|\alpha| \to \infty$, $\alpha \epsilon D_+$

and $\quad |f_n(\alpha)| < \frac{\text{const}}{n^2}$

so that

$$\lim_{|\alpha|\to\infty} \prod_{n=1}^{\infty} (1+f_n(\alpha)) = \prod_{n=1}^{\infty} \lim_{|\alpha|\to\infty} (1+f_n(\alpha)) = 1$$

Hence the infinite products in the expression for $K_+(\alpha)$ are $O(1)$ as $|\alpha|\to\infty$, $\alpha\epsilon D_+$. A similar result holds for the infinite products appearing in $K_-(\alpha)$. Thus

$$K_+(\alpha) = O(\alpha^2) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad D_+$$

$$K_-(\alpha) = O(\alpha^{-2}) \quad \text{as} \quad |\alpha| \to \infty \quad \text{in} \quad D_-$$

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TABLE 1. FREQUENCY AND WAVELENGTH BANDS CORRESPONDING TO INCIDENT WAVES WHICH ARE COMPLETELY REFLECTED

$s = 0.92$

MODEL SIZE Water Depth H = .50 Ft.				SCALE RATIO 80:1 H = 40 Ft							
Ice Thickness $h_o = h_1$ in.	Floe Length $2\pi/k$, ft	Incident Wave Length λ, ft	Incident Frequency ω, rad/sec	Ice Thickness $h_o = h_1$ ft	Floe Length $2\pi/k$, ft	Incident Wave Length λ, ft	Incident Frequency ω, rad/sec	Ice Thickness $h_o = h_1$ ft	Floe Length $2\pi/k$, ft	Incident Wave Length λ, ft	Incident Frequency ω, rad/sec
0.25	0.50	0-0.81	∞ -31.05	1.67	40	0-65	∞ -3.47	4.17	100	0-162	∞ -2.19
		1.08-1.24	23.27-20.27			86-99	2.61-2.27			216-248	1.65-1.44
	1.00	0-0.90	∞ -27.93		80	0-72	∞ -3.12		200	0-180	∞ -1.97
		1.12-1.16	22.44-21.67			90-93	2.51-2.43			224-232	1.59-1.54
		2.00-2.12	12.57-11.85			160-170	1.40-1.33			400-424	0.80-0.84
	3.00	-	-		240	-	-		600	-	-
1.00	0.50	0-1.72	∞ -14.61	6.67	40	0-138	∞ -1.64	16.67	100	0-344	∞ -1.04
	1.00	0-1.62	∞ -15.55		80	0-130	∞ -1.74		200	0-324	∞ -1.10
		2.14-2.49	11.77-10.10			171-199	1.32-1.13			428-498	0.83-0.71
	3.00	1.54-1.67	16.36-15.10		240	123-134	1.83-1.69		600	308-334	1.16-1.07
		1.89-1.91	13.34-13.20			151-153	1.49-1.48			378-382	0.94-0.93
2.00	0.50	0-2.16	∞ -11.64	13.33	40	0-173	∞ -1.30	33.33	100	0-433	∞ -0.82
	1.00	0-2.87	∞ - 8.77		80	0-230	∞ -0.98		200	0-574	∞ -0.62
	3.00	0-2.26	∞ -11.15		240	0-181	∞ -1.25		600	0-452	∞ -0.79
		2.30-2.64	10.95- 9.55			184-211	1.22-1.07			460-528	0.77-0.67
		3.40-3.48	7.41- 7.24			272-278	0.87-0.81			680-696	0.55-0.51
		5.65-6.34	4.46- 3.98			452-507	0.50-0.44			1130-1268	0.32-0.28

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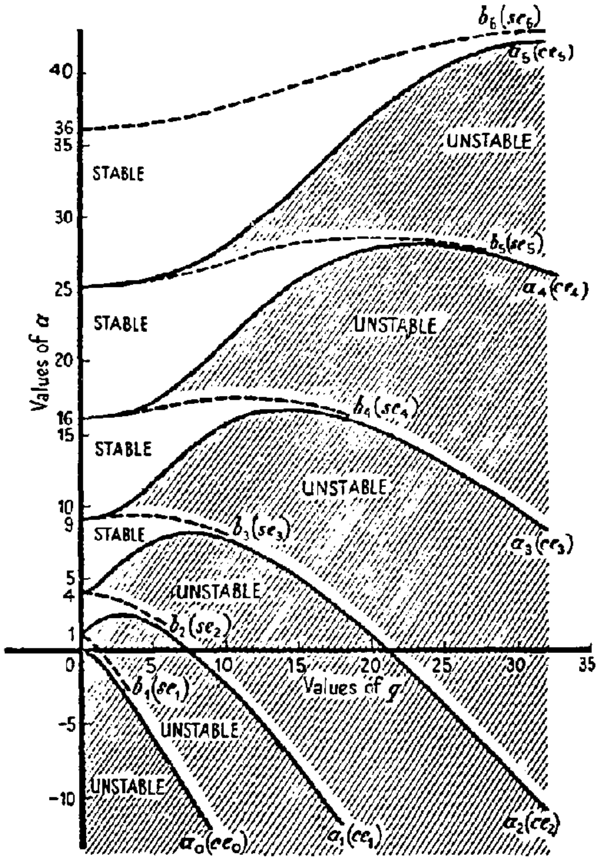

FIG. I. STABILITY DIAGRAM FOR SOLUTIONS OF MATHIEU'S DIFFERENTIAL EQUATION (REPRODUCED FROM McLACHLAN, THEORY AND APPLICATION OF MATHIEU FUNCTIONS, P. 40)

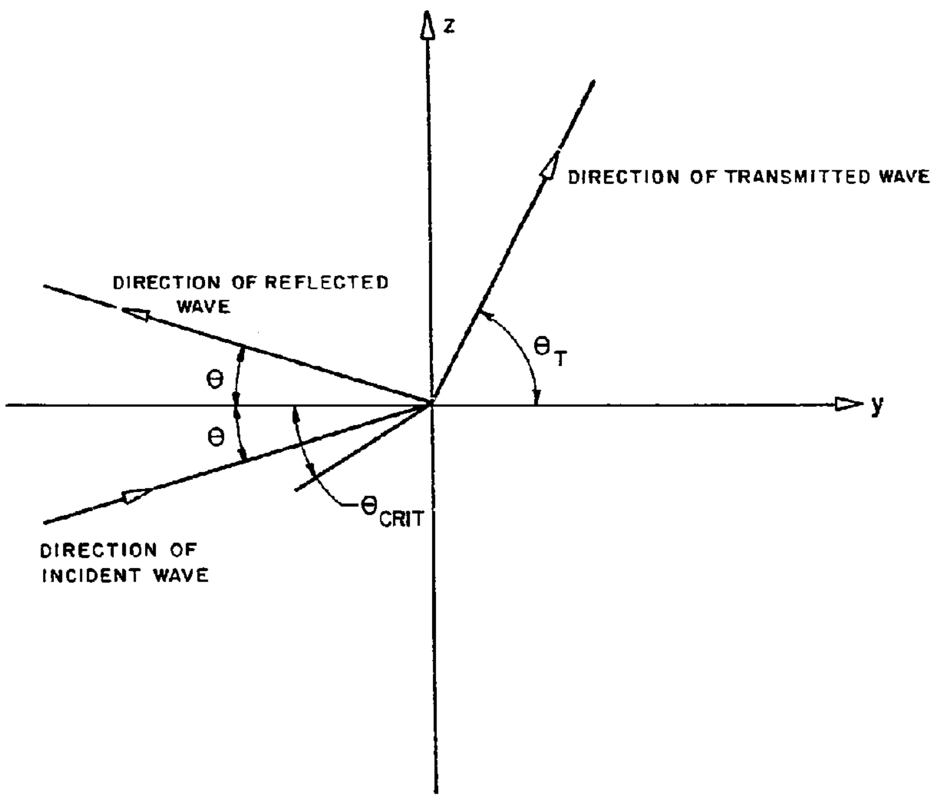

FIG. 2. REFLECTED AND TRANSMITTED WAVE FOR $\theta < \theta_{CRIT.}$

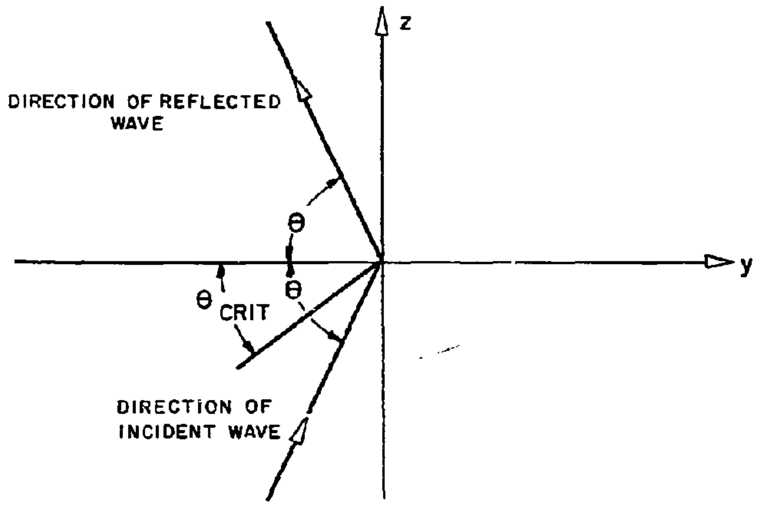

FIG. 3. COMPLETE REFLECTION, NO TRANSMITTED WAVE FOR $\theta > \theta_{CRIT}$

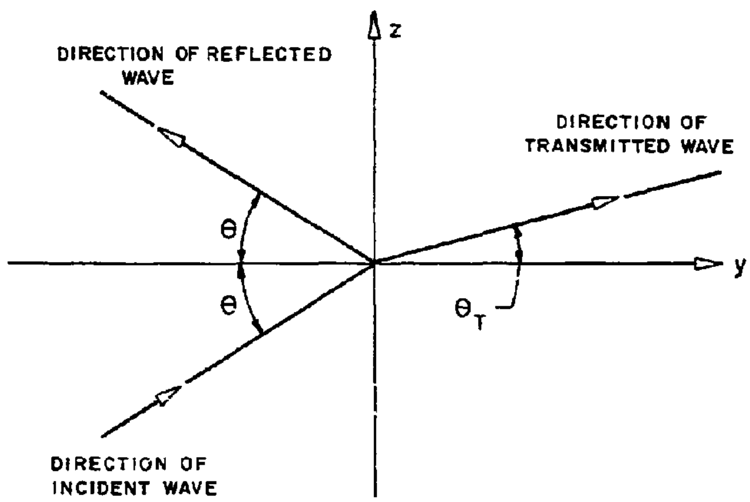

FIG. 4. ALWAYS TRANSMISSION FOR $\lambda > \lambda_{CRIT}$

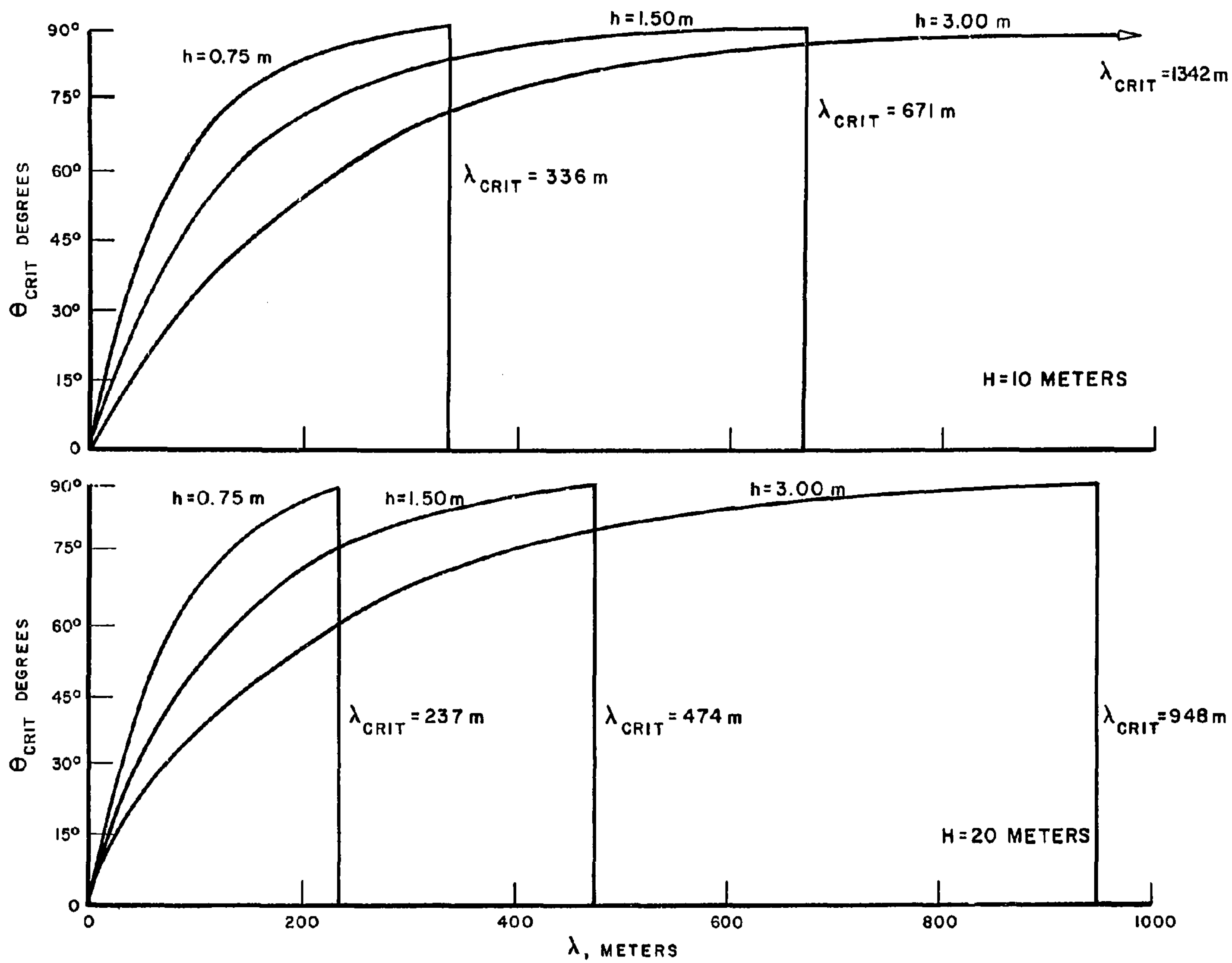

FIG. 5. RELATION BETWEEN CRITICAL ANGLE Θ_{CRIT} AND INCIDENT WAVE LENGTH λ

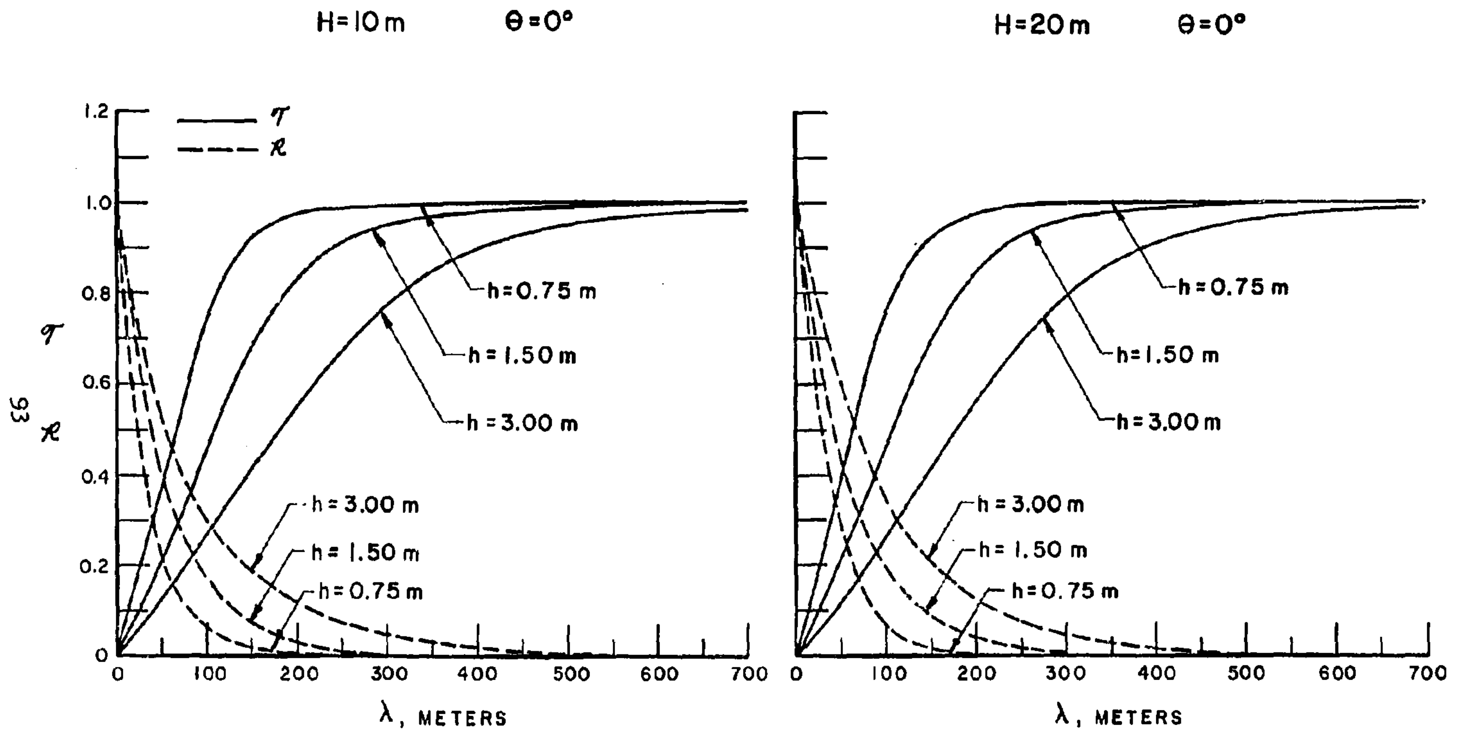

FIG. 6. REFLECTION $\mathcal{R}$ AND TRANSMISSION $\mathcal{T}$ COEFFICIENTS VS INCIDENT WAVE LENGTH λ

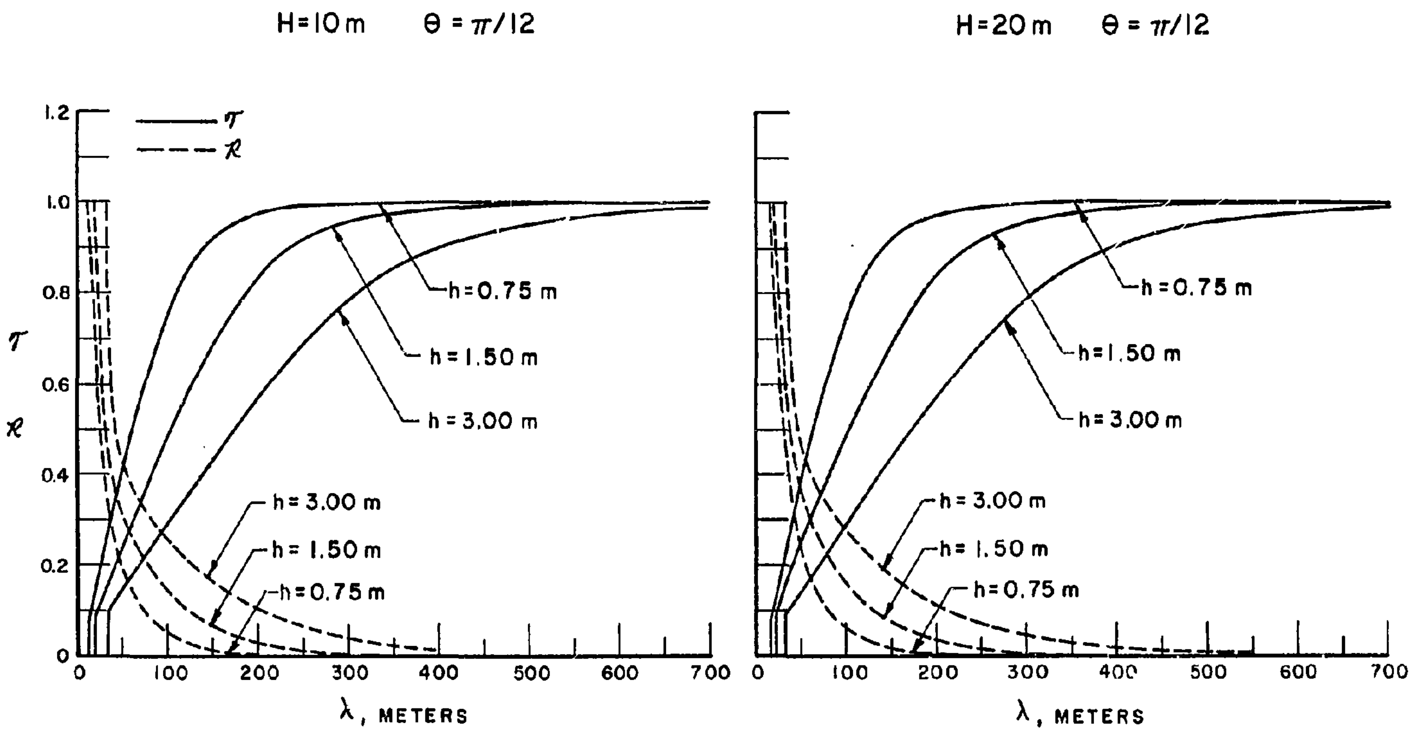

FIG. 7. REFLECTION $\mathcal{R}$ AND TRANSMISSION $\mathcal{T}$ COEFFICIENTS VS INCIDENT WAVE LENGTH λ

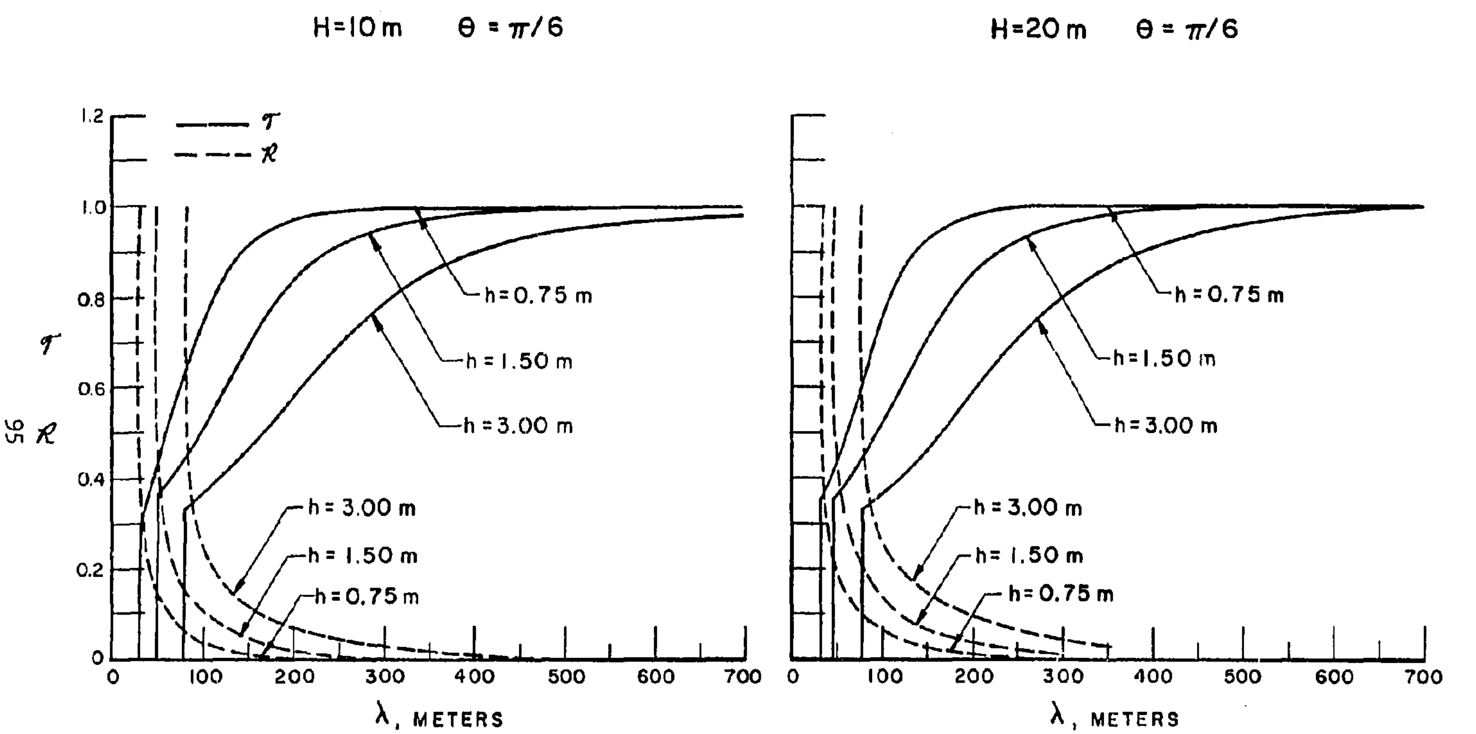

FIG. 8. REFLECTION ℛ AND TRANSMISSION 𝒯 COEFFICIENTS VS INCIDENT WAVE LENGTH λ

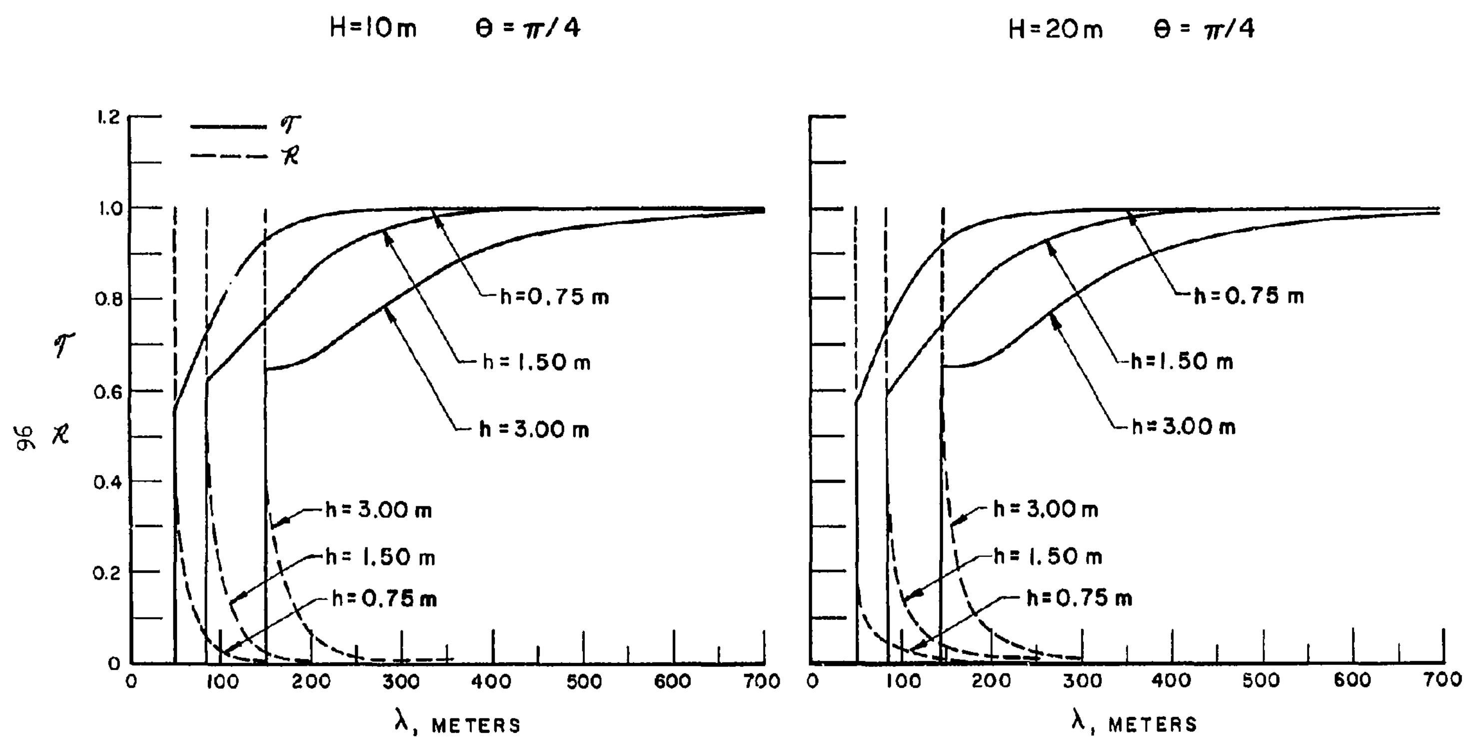

FIG. 9. REFLECTION ℛ AND TRANSMISSION 𝒯 COEFFICIENTS VS INCIDENT WAVE LENGTH λ

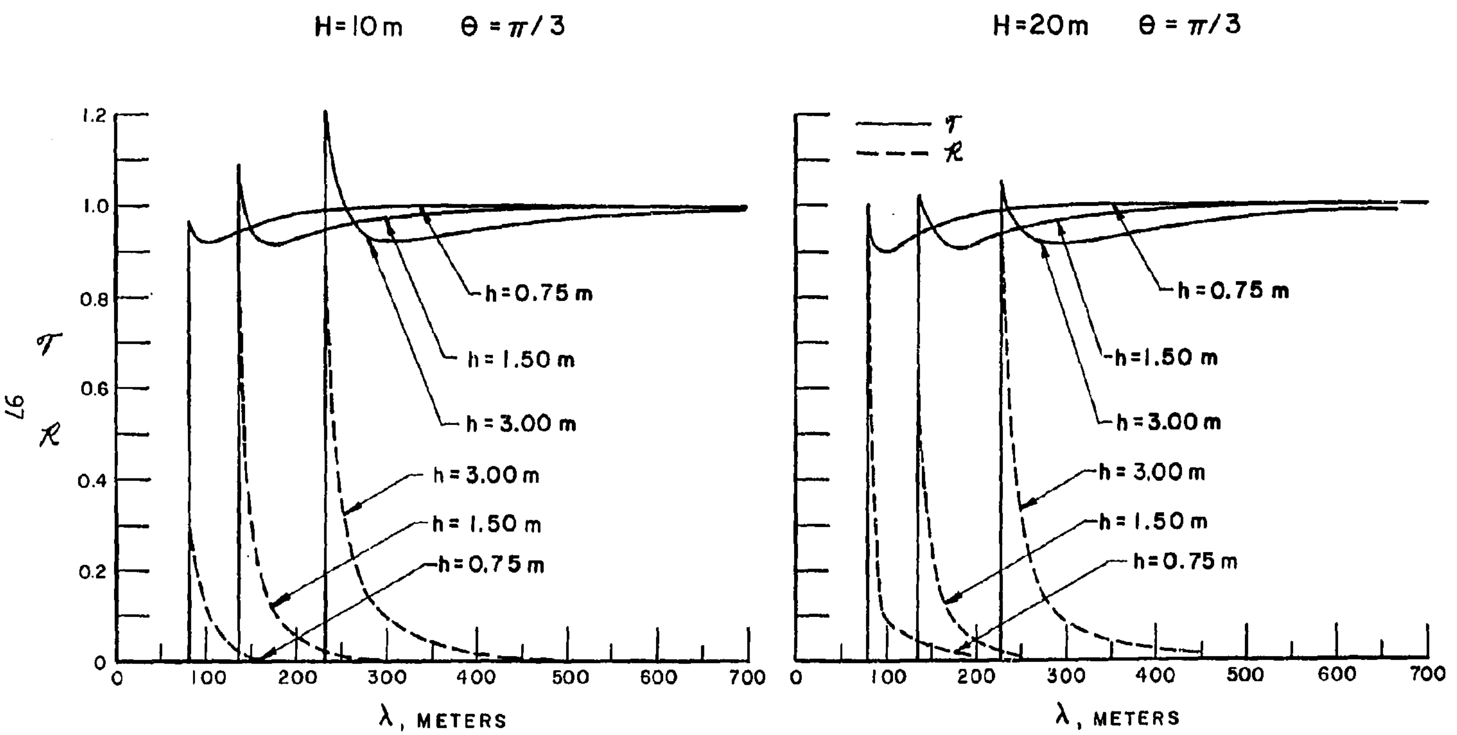

FIG. 10. REFLECTION ℛ AND TRANSMISSION 𝒯 COEFFICIENTS VS INCIDENT WAVE LENGTH λ

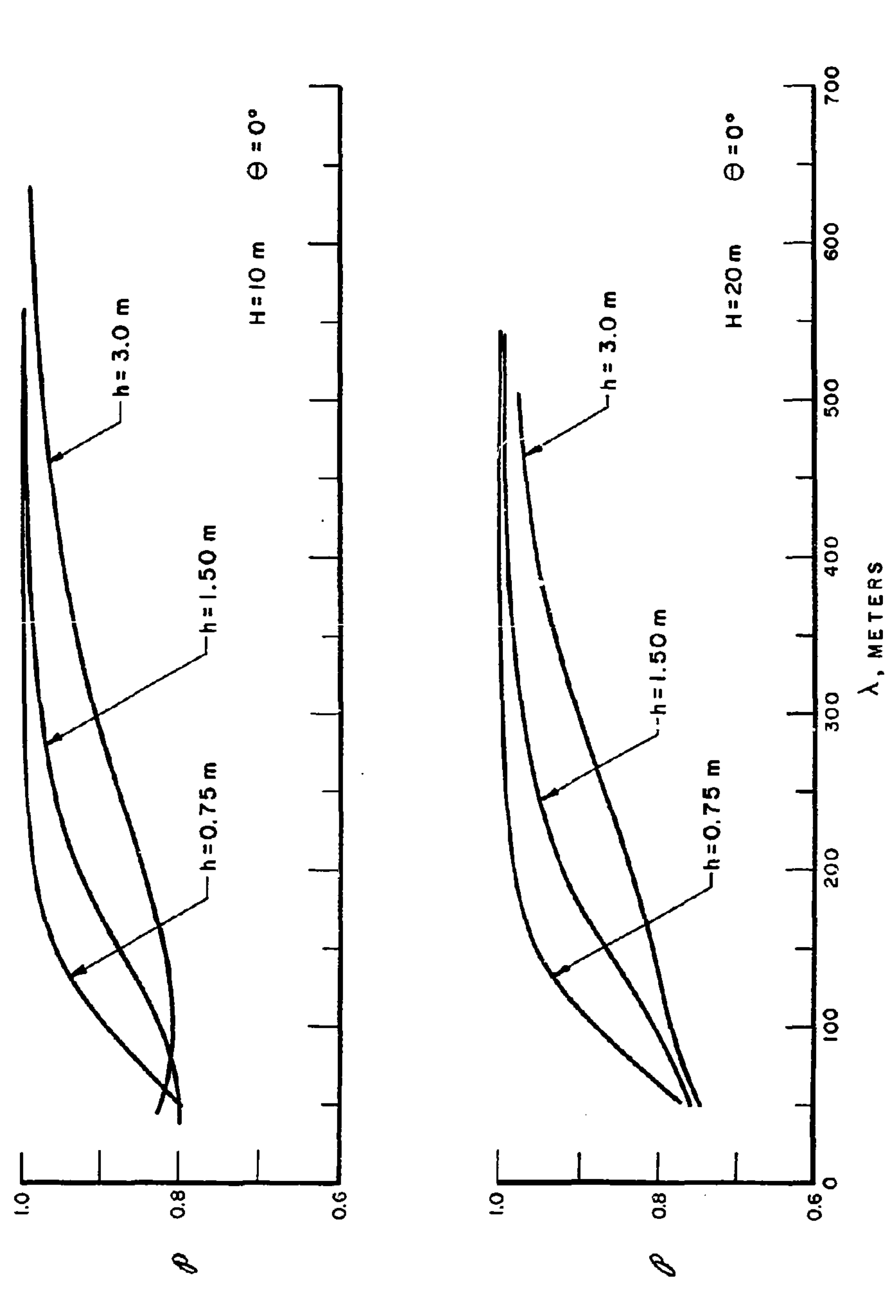

FIG. 11. BOTTOM PRESSURE FLUCTUATION β VERSUS INCIDENT WAVE LENGTH λ

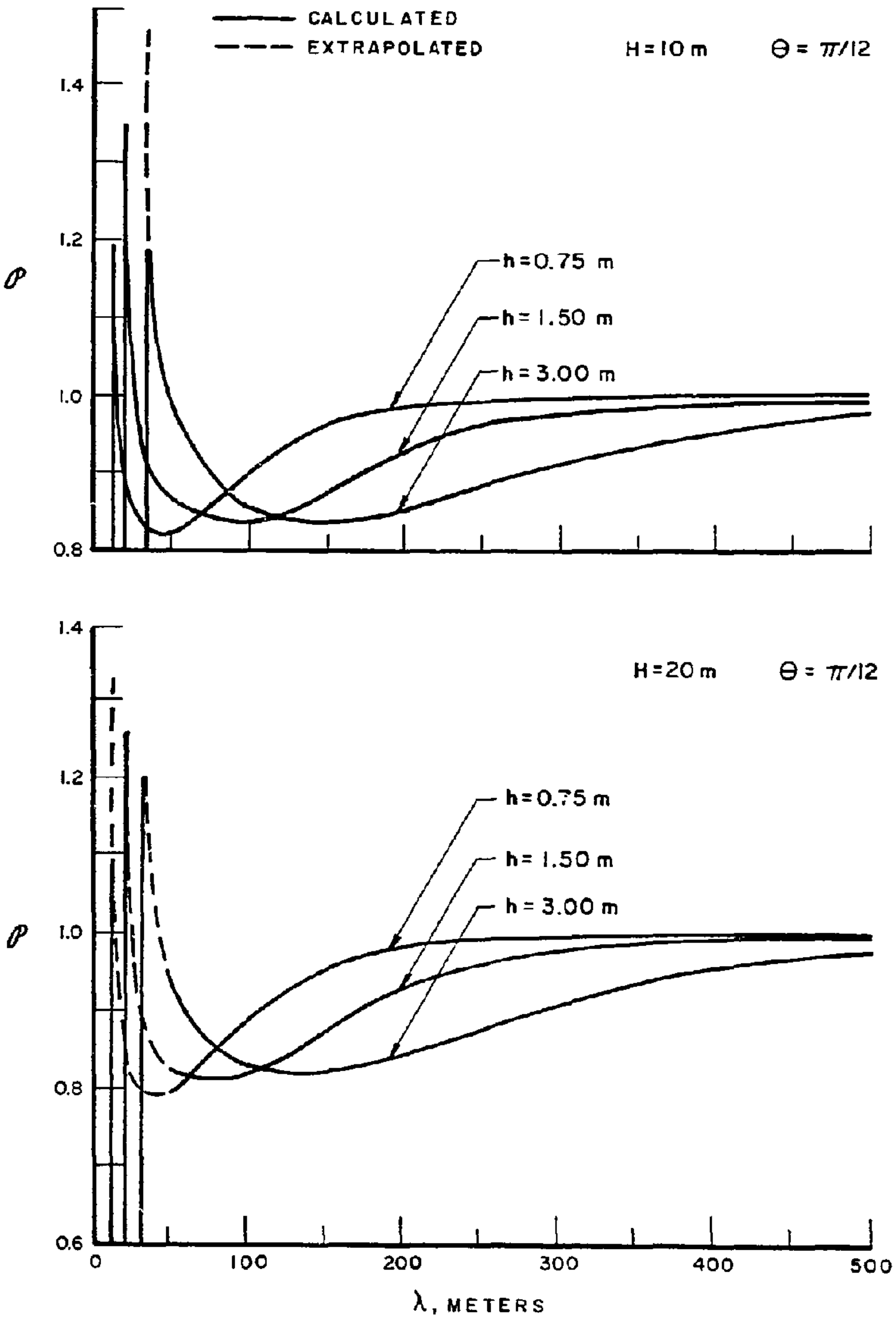

FIG. 12. BOTTOM PRESSURE FLUCTUATION $\mathcal{P}$ VERSUS INCIDENT WAVE LENGTH λ

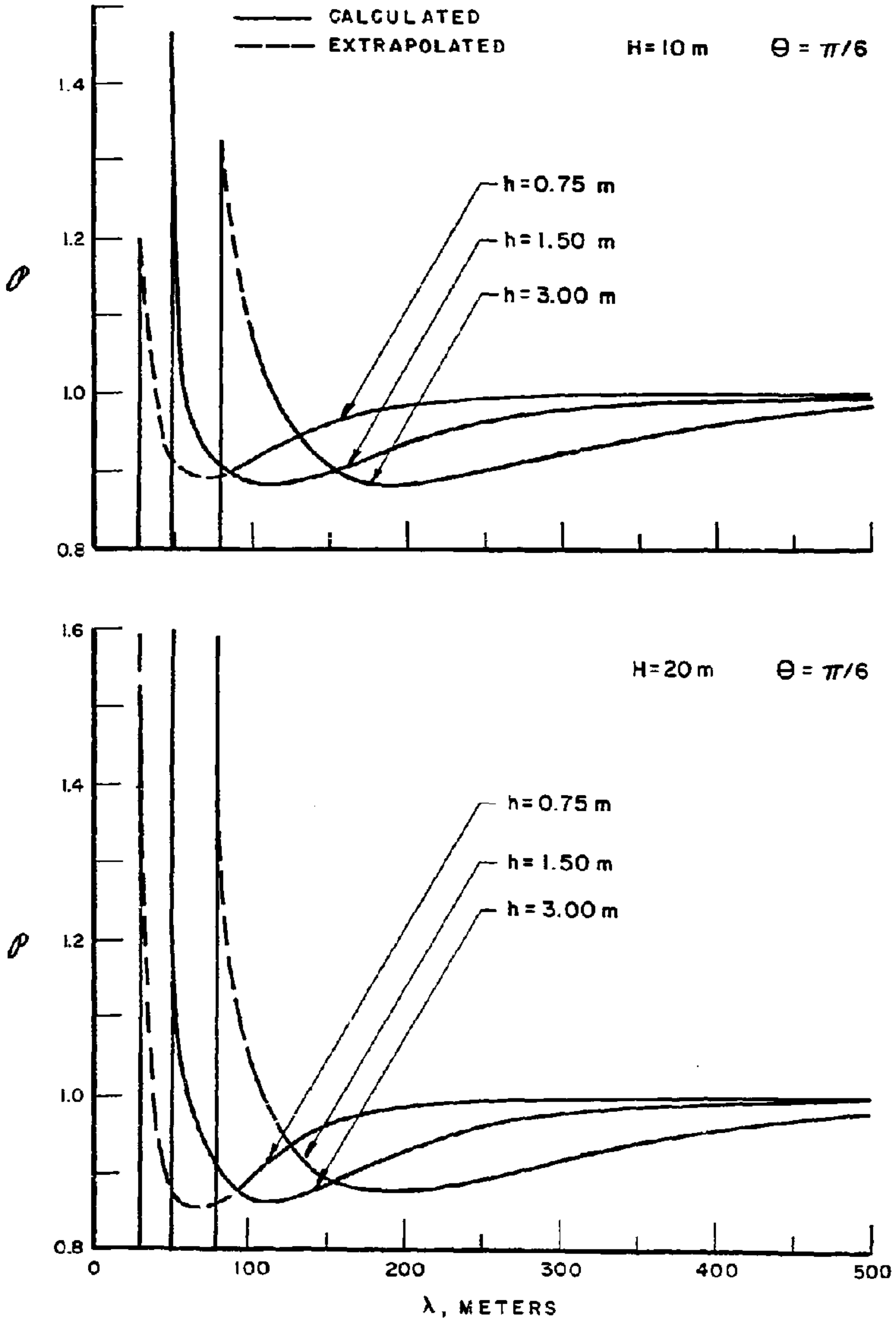

FIG. 13. BOTTOM PRESSURE FLUCTUATION $\mathcal{P}$ VERSUS INCIDENT WAVE LENGTH λ

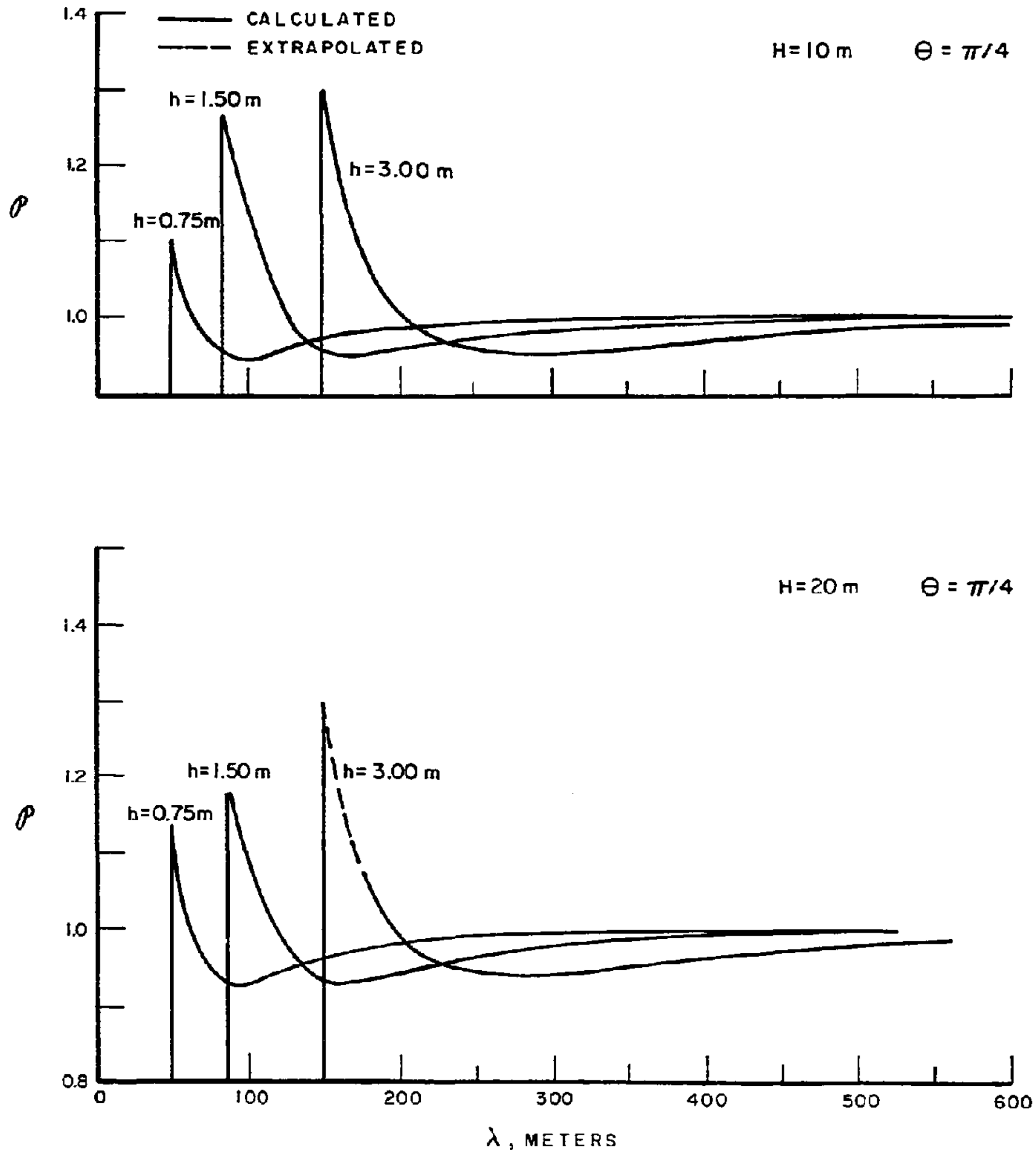

FIG. 14. BOTTOM PRESSURE FLUCTUATION 𝒫 VERSUS INCIDENT WAVE LENGTH λ

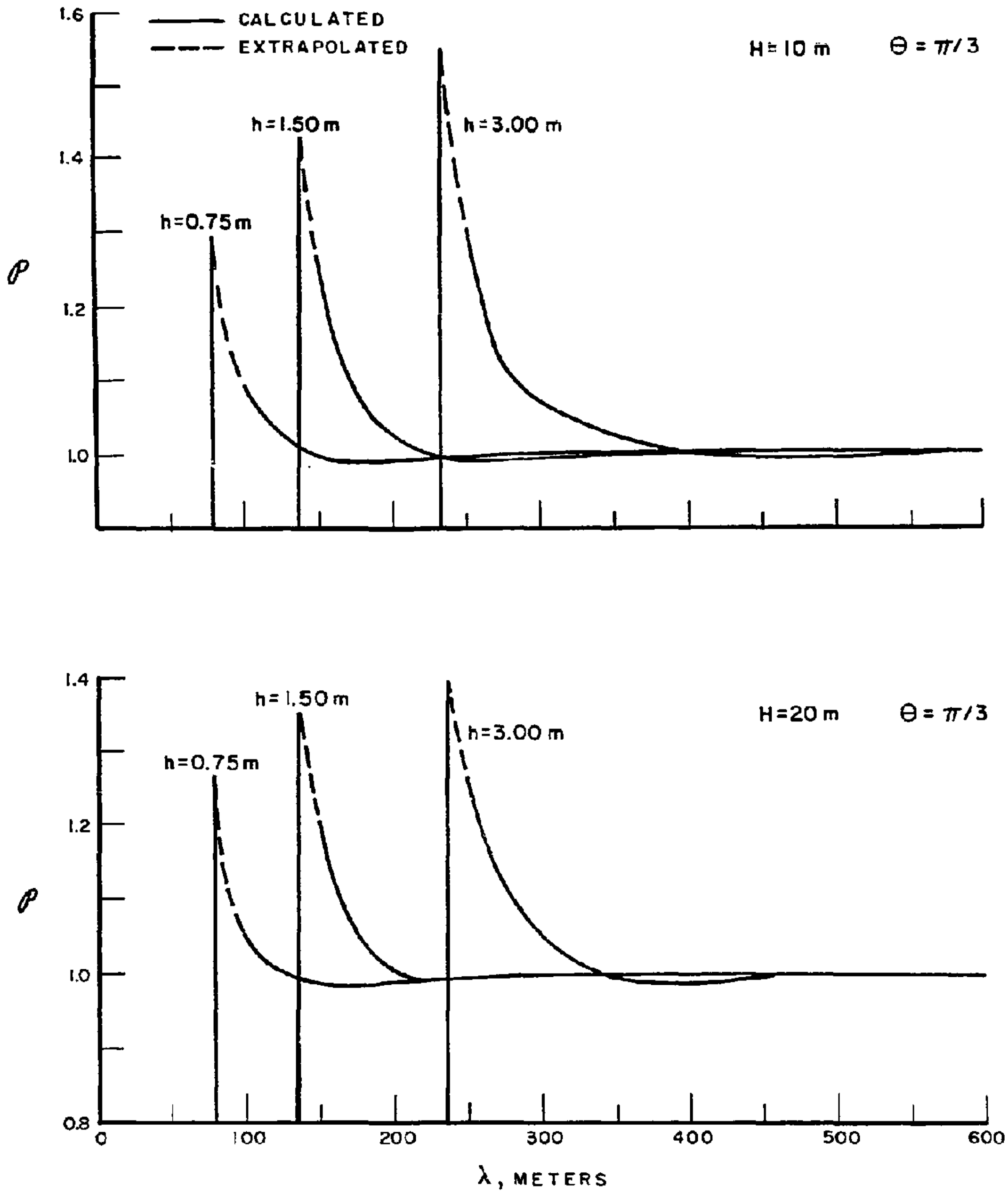

FIG. 15. BOTTOM PRESSURE FLUCTUATION 𝒫 VERSUS INCIDENT WAVE LENGTH λ

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13. ABSTRACT

Three models are examined to study the transmission of ocean waves through an ice-field. In each case the effect of ice thickness, water depth, and the wave-length and angle of incidence of the incoming ocean wave is considered. In Model I the ice is assumed to consist of floating non-interacting mass elements of varying thickness and the shallow-water approximation is utilized to simplify the equations. A simple cosine distribution varying in one direction only is assumed. In Model II the mass elements, of constant thickness, interact through a bending stiffness force so that the ice acts as a thin elastic plate. The mass elements are connected through a surface tension force in Model III so that the ice is simulated by a stretched membrane. In both Models II and III the full linearized equations are solved. Because of the complexity of the resulting analysis, calculations of the reflection and transmission coefficients, and the pressure under the ice, are made in Model II on the basis of the shallow water approximation.

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